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一类带调和势的非线性Schrödinger方程的解

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【摘要】研究了一类带调和势的Schrödinger方程的解，运用能量守衡定律和质量守衡定律以及利用矢量分析的知识，引入积分不等式和解微分不等式的方法，得到了初值满足一定条件的柯西问题的解会在有限的时间里发生爆破的结论。由于所讨论方程更具有一般性，从而推广了已有的结论，所得到结论也可以对能量和质量的集中现象作进一步解释。

关　键　词　调和势; 非线性Schrödinger方程; 爆破; 能量守衡

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Solutions to a Class of Nonlinear Schrödinger Equation with Harmonic Potential

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Abstract　In the paper, a class of Schrödinger equation with harmonic Potential, which concerns Bose-Einstin condensates, is investigated. Now that the equation of Bose-einstin condensates describes lot of phenomena, that we research it has special significance. With the help of using energy conservative law and quality conservative law as well as knowledge of vector analysis, integral and differential inequality, we prove that the solution to the Cauchy's problem will blow up in finite time in case initial value satisfy with some conditions. As the equation in the paper is more general, we have got extensive conclusion, by means of which we may deeply understand the aggregative phenomena on energy and quality.

Key words　nonlinear Schrödinger equation;　harmonic potential;　blow up;　energy conservative

Since Bose-Einstein condensate equations have recently caused our extensive attention, a class of nonlinear schrödinger equations with harmonic potential

$$ij_t = -(1/2)\Delta j + (1/2)|x|^2 j - a|j|^2 j - b|j|^4 j$$

has been discussed and some existent traits and instabilities of solution have been got in Refs. [1～10].

On that base a class of more general nonlinear Schrödinger equations with harmonic potential

$$ij_t = -(1/2)\Delta j + (1/2)g(|x|^2)j + f(|j|^2)j \tag{1}$$

$$j(0,x) = j_0 \tag{2}$$

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where $f(t)$ is non-negative continuous and monotonically decreasing function in $[0,+\infty)$, $g(t)$ is monotonically increasing and non-negative derivable function in $[0,+\infty)$, $t \geq 0$, $x \in R^n$, i is imaginary unit and Δ is Laplace operator, will be investigated by means of employing analogous to the method in Refs. [7～10] and we get the instability of solution to the nonlinear Schrödinger equations under given condition.

1 Preliminary knowledge and Lemma

We introduce energy space: $H := \{j \in H^1, \int_{R^n} |x|^2 |j|^2 \mathrm{d}x < \infty\}$. Let $J(t) = \int_{R^n} |x|^2 |j|^2 \mathrm{d}x$, $E(t)$ denote energy function. By using the result of Refs. [5]and [6], we get the local existence of the solution to the Eqs. (1) , (2) with initial value condition. In convenience we denote $\int_{R^n} = \int$. Now we formalize it.

Theorem 1 If $j_0 \in H$, then the Eqs. (1) , (2) has a unique solution j in maximum time interval $[0,T)$, which satisfies with $j \in C([0,T);H)$.

Lemma 1 If j is the solution to the Eqs. (1) , (2) and $j_0 \in H$, then we get below formulas:

(1) $\int |j|^2 \mathrm{d}x = \int |j_0|^2 \mathrm{d}x$;

(2) $E(t) = \int [(1/2)|\nabla j|^2 + (1/2) g(|x|^2)|j|^2 + F(|j|^2)]\mathrm{d}x = E(0)$;

(3) $J'(t) = 2\,\mathrm{Im}\int x\bar{j}\,\nabla j\,\mathrm{d}x = -2\,\mathrm{Im}\int x j \nabla \bar{j}\,\mathrm{d}x$;

(4) $J''(t) = 2\{\int |\nabla j|^2 \mathrm{d}x - \int |x|^2 g'(|x|^2)|j|^2 \mathrm{d}x + n\int [f(|j|^2)|j|^2 - F(|j|^2)]\mathrm{d}x\}$, where $F(x) = \int_0^x f(t)\mathrm{d}t$.

Proof of the formula (1): If

$$m(j) = m(t) = \int |j|^2 \mathrm{d}x$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int |j|^2 \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int j\bar{j}\,\mathrm{d}x = 2\,\mathrm{Re}\int \bar{j} j_t \mathrm{d}x$$

By multiplying the Eq. (1) by $(-i\bar{j})$ and then integrating x on R^n

$$\int j_t \bar{j}\,\mathrm{d}x = -i\int [(1/2)|\nabla j|^2 + (1/2) g(|x|^2)|j|^2 + f(|j|^2)|j|^2]\mathrm{d}x \tag{3}$$

So $\mathrm{Re}\int j_t \bar{j}\,\mathrm{d}x = 0$, i. e. $\frac{\mathrm{d}}{\mathrm{d}t} m(t) = 0$. Hence $\int |j|^2 \mathrm{d}x = \int |j_0|^2 \mathrm{d}x$.

Proof of the formula (2): By multiplying Eq.(1) by $\bar{j}_t$

$$i j_t \bar{j}_t = -(1/2)\Delta j \bar{j}_t + (1/2) g(|x|^2) j \bar{j}_t + f(|j|^2) j \bar{j}_t \tag{4}$$

By means of conjugating two sides of Eq.(4), we get

$$-i\bar{j}_t j_t = -(1/2)\Delta \bar{j} j_t + (1/2) g(|x|^2)\bar{j} j_t + f(|j|^2)\bar{j} j_t \tag{5}$$

By Eq. (4)+Eq. (5) and then integrating x on R^n, it changes

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int [(1/2)|\nabla j|^2 + (1/2) g(|x|^2)|j|^2 + F(|j|^2)]\mathrm{d}x$$

Let $E(t) = \int [(1/2)|\nabla j|^2 + (1/2) g(|x|^2)|j|^2 + F(|j|^2)]\mathrm{d}x$, i.e. $E(t) = E(0) = \mathrm{const.}$

Proof of the formula (3): As to the process of the proof we refer to Ref. [8].

Proof of the formula (4): By multiplying Eq.(1) by $2x\nabla\bar{j}$

$$2xi\nabla \bar{j} j_t = -x\nabla\bar{j}\Delta j + x\nabla\bar{j} g(|x|^2) j + 2x\nabla\bar{j} f(|j|^2) j \tag{6}$$

We take real part from the above equation and then integrate x on R^n

$$\int \mathrm{Re}(2xi\nabla\bar{j} j_t)\mathrm{d}x = \int \mathrm{Re}(-x\nabla\bar{j}\Delta j)\mathrm{d}x + \int \mathrm{Re}[x\nabla\bar{j} g(|x|^2) j]\mathrm{d}x + \int \mathrm{Re}[2x\nabla\bar{j} f(|j|^2) j]\mathrm{d}x \tag{7}$$

In light of the Ref. [8] $\int \mathrm{Re}(2xi\nabla\bar{j} j_t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Im}\int x\bar{j}\nabla j\,\mathrm{d}x + n\,\mathrm{Im}\int \bar{j} j_t \mathrm{d}x.$

According to Eq.(3)

$$\int \mathrm{Re}(2xi\nabla\bar{j}\,j_t)\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\int x\bar{j}\,\nabla j\,\mathrm{d}x-n\int[(1/2)|\nabla j|^2+(1/2)g(|x|^2)|j|^2+f(|j|^2)|j|^2]\mathrm{d}x \tag{8}$$

From partial integral formula (see Ref.[8]), we have

$$\int \mathrm{Re}(-x\nabla\bar{j}\Delta j)\mathrm{d}x=-[(n-2)/2]\int|\nabla j|^2\mathrm{d}x \tag{9}$$

Since

$$\int \mathrm{Re}[x\nabla\bar{j}\,g(|x|^2)j]\mathrm{d}x=\mathrm{Re}\int xg(|x|^2)j\nabla\bar{j}\mathrm{d}x=-\mathrm{Re}\int\bar{j}\,\nabla(xg(|x|^2)j)\mathrm{d}x=$$

$$-\int[ng(|x|^2)+2|x|^2g'(|x|^2)]|j|^2\mathrm{d}x-\mathrm{Re}\int xg(|x|^2)\bar{j}\,\nabla j\,\mathrm{d}x$$

So

$$\int \mathrm{Re}[x\nabla\bar{j}\,g(|x|^2)j]\mathrm{d}x=-(1/2)\int[ng(|x|^2)+2|x|^2g'(|x|^2)]|j|^2\mathrm{d}x \tag{10}$$

According to Ref. [10]

$$\int \mathrm{Re}[2x\nabla\bar{j}f(|j|^2)j]\mathrm{d}x=-n\int F(|j|^2)\mathrm{d}x \tag{11}$$

By taking Eqs.(8～11) into Eq.(7), it is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\int x\bar{j}\,\nabla j\,\mathrm{d}x=\int|\nabla j|^2dx-\int|x|^2g'(|x|^2)|j|^2\mathrm{d}x+n\int[f(|j|^2)|j|^2-F(|j|^2)]\mathrm{d}x$$

Hence $J''(t)=2\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\int x\bar{j}\,\nabla j\,\mathrm{d}x=2\{\int|\nabla j|^2\mathrm{d}x-\int|x|^2g'(|x|^2)|j|^2\mathrm{d}x+n\int[f(|j|^2)|j|^2-F(|j|^2)]\mathrm{d}x\}$

Lemma 2 Let $w(x)=nxf(x)-(n+2)\int_0^x f(t)\mathrm{d}t$. If $f(x)$ is non-negative continuous and monotonically decreasing function in $[0,+\infty)$, then $w(x)\le 0$ in $[0,+\infty)$.

Proof　Since by means of integral inequality we may simply prove it, the proof is omitted.

2　Result and Proof

Theorem 2 If $j_0\in H$ and j is the solution to the Eqs. (1) , (2) and when initial value j_0 satisfies with one of the below conditions:

H_1　$E(0)<0$;

H_2　$E(0)=0,\mathrm{Im}\int xj_0\nabla\bar{j}_0dx>0$;

H_3　$E(0)>0,\mathrm{Im}\int xj_0\nabla\bar{j}_0\mathrm{d}x\ge[2E(0)J(0)]^{1/2}$;

then there is a finite time T which lets

$$\lim_{t\to T}\|\nabla j\|_{L^2}=\infty$$

Proof　If T is infinite and in terms of the energy formula (2), we have

$$2\int|\nabla j|^2\mathrm{d}x=4E(0)-2\int g(|x|^2)|j|^2\mathrm{d}x-4\int F(|j|^2)\mathrm{d}x \tag{12}$$

Put Eq.(12) into the formula (4), we get

$$J''(t)=2\{\int|\nabla j|^2\mathrm{d}x-\int|x|^2g'(|x|^2)|j|^2\mathrm{d}x+n\int[f(|j|^2)|j|^2-F(|j|^2)]\mathrm{d}x\}=$$

$$4E(0)-2\int[g(|x|^2)+|x|^2g'(|x|^2)]|j|^2\mathrm{d}x+\int[2n|j|^2f(|j|^2)-(2n+4)F(|j|^2)]\mathrm{d}x$$

i.e. $$J''(t)\le 4E(0)+2\int[n|j|^2f(|j|^2)-(n+2)F(|j|^2)]\mathrm{d}x\qquad(0\le t<\infty) \tag{13}$$

By reason of $|j|^2\ge 0$ and in the light of Lemma 2, Eq.(13) may change

$$J''(t)\le 4\mathrm{E}(0)\qquad(0\le t<\infty) \tag{14}$$

According to Eq.(14) we have

$$J(t) \leq J(0) + J'(0)t + 2E(0)t^2 \qquad (0 \leq t < \infty) \tag{15}$$

Since

$$J(t) = \int |x|^2 |j|^2 \mathrm{d}x \geq 0, \quad J(0) = \int |x|^2 |j_0|^2 \mathrm{d}x > 0$$

and

$$J'(0) = -2\,\mathrm{Im} \int x j_0 \nabla \bar{j}_0 \mathrm{d}x > 0$$

so under one of the condition H_1, H_2, H_3 there is a finite time T^* which satisfies with

$$\lim_{t \to T^*} J(t) = 0$$

That is

$$\lim_{t \to T^*} \int |x|^2 |j|^2 \mathrm{d}x = 0 \tag{16}$$

Since the (16) contradicts the Lemma 1, hence T is finite.

Therefore there is finite time T which lets

$$\mathrm{Lim}_{t \to T} \|\nabla j\|_{L^2} = \infty$$

By means of it we may understand the aggregative phenomena on energy and quality.

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