Another Generalized Nonlinear Diagonal Dominance

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Abstract Another concept of generalized diagonal dominance for nonlinear functions (i.e. quasi-generalized nonlinear diagonal dominance) is introduced. This concept includes as special case generalized nonlinear diagonal dominance, possess much of important properties of generalized nonlinear diagonal dominance, easy to judge a nonlinear function is quasi-generalized nonlinear diagonally dominant if its derivative is a generalized diagonally dominant matrix on an convex set in $\boldsymbol{R}^n$.

Key words nonlinear functions; diagonal dominance; injection; derivative

非线性对角占优性

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【摘要】提出了另一种非线性对角占优性(即拟对角占优性)，推广了现有非线性对角占优性，同时，拟对角占优性具有许多良好的性质，可以方便地利用其导函数是*H*矩阵去判定一个非线性函数是拟对角占优的。研究了拟对角占优函数的性质，并对判定非线性广义对角占优性的公开问题给出了反例。

关 键 词 非线性函数; 对角占优; 单射; 导数

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In order to generalize diagonally dominant matrix to nonlinear case, Jorge Moré introduced the concepts of (strictly) diagonal dominance, and (weakly) $\boldsymbol{W}$ –diagonal dominance for nonlinear functions, and applied these concepts to Gauss-Seidel iterations[1]. Ref.[2] introduced the concept of nonlinear block diagonal dominance, discussed its relations to strictly diagonally dominant functions and certain *M*-functions and proved the global convergence of block asynchronous SOR-methods for fining zeros of nonlinear functions. In order to generalize ***H***-matrix to nonlinear case Andreas Frommer introduced the concept of generalized diagonal dominance for nonlinear functions, with this concept they obtained a quite far-reaching result on the global convergence of asynchronous iterative methods for fining zeros of nonlinear functions[3]. But it seems to be difficult to determine a function is generalized diagonally dominant even though its derivative is a ***H***-matrix. In fact, it is still an open question that even if F is continuously differentiable –whether $F'(x)$ being generalized diagonally dominant for all $x\in Q$ is sufficient for F being generalized diagonally dominant on Q. So it seems to be of some interesting to introduced another concept generalizing ***H***-matrix to nonlinear case, we name as quasi-generalized diagonally

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dominant functions, one can easy to determine a nonlinear function to be quasi-generalized diagonally dominant functions by its derivative being a ***H***-matrix, in this paper we devote to establish fundamental properties and to clarify the relationship between the different classes of functions.

1 Review of the Linear Case

We use the symbol *N* to denote the set $N := \{1, 2, \cdots, n\}$ for all natural number *n*. The ordering "$\leq$" and absolute value $|\bullet|$ on R^n and $R^{n\times n}$ be understood component wise as usual. In addition, we write "<" in R^n or $R^{n\times n}$ if we have strict inequality for all components.

Throughout this paper $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in R^{n\times n}$ is assumed to be a nonsingular matrix.

Definition 1 ***A*** is call generalized diagonally dominant matrix (namely ***H***-matrix) if there exists a vector $\boldsymbol{u} \in R^n, \boldsymbol{u} > 0$ such that: $|\boldsymbol{a}_{ii}\boldsymbol{u}_i| > \sum_{j=1,\neq i}^{n} |\boldsymbol{a}_{ij}\boldsymbol{u}_j|$ **,** $i = 1, 2, \cdots, n$.

The class of generalized diagonally dominant matrices contains several important classes of matrices arising in applications, such as strictly diagonally dominant matrices, ***W*** – diagonally dominant matrices, and ***M***-matrices

2 Quasi-Generalized Nonlinear Diagonal Dominance

Given a function $F: R^n \to R^n$, let F_i denote the *i*th component of *F*, i.e., $F = (F_1x, F_2x, \cdots, F_nx)^T$ as a motivation to what will follow, let us consider the affine function $F: x \in R^n$ a $\boldsymbol{A}x - b$ with $\boldsymbol{A} \in R^{n\times n}$, $b \in R^n$. A straightforward calculating then shows that ***A*** is a generalized diagonally dominant if and only if there exists a vector $\boldsymbol{u} \in R^n$, $\boldsymbol{u} > 0$ such that the following implication is true for $i = 1, 2, \cdots, n$:

$$x \neq y, F_i x = F_{iy} \Rightarrow \left|\tfrac{1}{u_i}(y_i - x_i)\right| < \max_{j=1}^{n} \left|\tfrac{1}{u_j}(y_j - x_j)\right| \tag{1}$$

The clue to the generalization to the nonlinear case now is to allow the "weighting" through the factors u_i in Eq. (1) to be the nonlinear and to dependent on both *x* and *y*.

Definition 2[3] $F: Q \to R^n$ is called generalized nonlinear diagonally dominant (on *Q*), if for every $x \in Q$ there exists a function $U^x: Q \to R^n$ such that: 1) U^x is diagonal, continuous and strictly istone. 2) $U^x x = x$. 3) for $i = 1, 2, \cdots, n$, we have $x \neq y, F_i x = F_i y \Rightarrow |U_i^x y_i - x_i| < \|U^x y - x\|_\infty$.

We now turn to our definition of quasi-generalized nonlinear diagonal dominance.

Definition 3 $F: Q \to R^n$ is called quasi-generalized nonlinear diagonally dominant (on *Q*), if for every pair $(x, y) \in Q \times Q$ there exists a function $V^{(x,y)}: Q \to R^n$ such that: 1) $V^{(x,y)}$ is diagonal, continuous and strictly istone. 2) $V^{(x,y)}x = x$. 3) for $i = 1, 2, \cdots, n$, we have

$$x \neq y, F_i x = F_i y \Rightarrow |V^{(x,y)} y_i - x_i| < \|V^{(x,y)} y - x\|_\infty \tag{2}$$

Clearly, taking $V_i^{(x,y)} t = u_i^{-1}(t - x_i) + x$ then Definition 1 results in Eq.(2) we thus have shown proof for the following Theorem:

Theorem 1 Let *F* be affine, *Fx*=***A****x*-*d* with $\boldsymbol{A} \in R^{n\times n}$, $d \in R^n$. Then *F* is quasi-generalized nonlinear diagonally dominant on R^n (in sense of Definition 5) if ***A*** is ***H***-matrix.

We now investigate some basic properties of quasi-generalized nonlinear diagonally dominant functions.

Theorem 2 Let $F: Q \to R^n$**,** then the following assertions are true.

1) If F is generalized diagonally dominant (on Q), then F is quasi-generalized nonlinear diagonally dominant (on Q). 2) If F is called strictly diagonally dominant (on Q), then F is quasi-generalized nonlinear diagonally dominant (on Q). 3) Let F is weakly Ω- diagonally dominant (on Q) in sense of Ref.[1] and continuous on Q, then F is quasi-generalized nonlinear diagonally dominant on every compact rectangle $\tilde{Q} \subseteq Q$. 4) Let F be an *M*-function, i.e., F is off-diagonally antitone and inverse isotone on *Q*. then F is

quasi-generalized nonlinear diagonally dominant, provided $Q = R^n$ and F is continuous and subjective.

Theorem 3 Let $F: Q \to R^n$, Then F is quasi-generalized nonlinear diagonally dominant (on Q) if for all $(x, y) \in Q \times Q$ there exists a function $V^{(x,y)}: Q \to \tilde{Q}$ where $V^{(x,y)}$ is diagonal, continuous, strictly istone, $V^{(x,y)}x = x$ such that $F \circ (V^{(x,y)})^{-1}$ is strictly diagonally dominant (on $\tilde{Q}$).

Proof If $F \circ (V^{(x,y)})^{-1}$ is strictly diagonally dominant (on $\tilde{Q}$), for $i = 1, 2, \cdots, n$ and $\forall (x, y) \in Q \times Q$, $x \neq y$ such that $F_i x = F_i y$, denote $\bar{x} = V^{(x,y)}x$, $\bar{y} = V^{(x,y)}y$, then $\bar{x} \in \tilde{Q}$, $\bar{y} \in \tilde{Q}$, $\bar{x} \neq \bar{y}$ and $[F \circ (V^{(x,y)})^{-1}]_i \bar{x} = F_i x$, $[F \circ (V^{(x,y)})^{-1}]_i \bar{y} = F_i y$. Therefore:

$$[F \circ (V^{(x,y)})^{-1}]_i \bar{x} = [F \circ (V^{(x,y)})^{-1}]_i \bar{y} \tag{3}$$

by strictly diagonal dominance of $F \circ (V^{(x,y)})^{-1}$, Eq.(3) implies $|\bar{x}_i - \bar{y}_i| < \|\bar{x} - \bar{y}\|_\infty$ i.e.,

$$\left|V_i^{(x,y)} y_i - x_i\right| < \left\|V^{(x,y)} y - x\right\|_\infty$$

We have shown that F is quasi-generalized nonlinear diagonally dominant (on Q).

Theorem 4 If Q and $\tilde{Q}$ are two rectangle in R^n, $F: Q \to R^n$ is quasi-generalized diagonally dominant (on Q), $V: \tilde{Q} \to Q$ be a diagonal, continuous and strictly istone, then $\tilde{F}: \tilde{Q} \to R^n$, $x \mapsto F(V(x))$ is quasi-generalized nonlinear diagonally dominant (on $\tilde{Q}$).

The proof for Theorem 4 is similar to that of Theorem 3, we therefore refrain from repeating the details.

Theorem 5 Let $F: Q \to R^n$ be quasi-generalized nonlinear diagonally dominant, then F is injective.

Proof Assume $Fx = Fy$ for some $x, y \in Q$, $x \neq y$. Then we have for $i = 1, 2, \cdots, n$,

$$\left|V^{(x,y)} y_i - x_i\right| < \left\|V^{(x,y)} y - x\right\|_\infty$$

Whence $\left\|V^{(x,y)} y - x\right\|_\infty < \left\|V^{(x,y)} y - x\right\|_\infty$, which is absurd.

Theorem 6 Let $F: Q \to R^n$ be continuous and quasi-generalized nonlinear diagonally dominant. Then for any $i \in \{1, 2, \cdots, n\}$ the function $y_{ii}^{x}: Q_i \to R$ is either strictly isotone for all $x \in Q$ or strictly antitone for all $x \in Q$.

Proof By the continuity of F we know that for $i \in \{1, 2, \cdots, n\}$ every function y_{ii}^{x} is continuous. We start by showing that $y_{ii}^{x}: Q_i \to R$ is injective.

Suppose $y_{ii}^{x} s = y_{ii}^{x} t$, $s \neq t$. Introducing the notation: $y := (x_1, \cdots, x_{i-1}, s, x_{i+1}, \cdots, x_n)^T \in Q$, $z := (x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n)^T \in Q$. we thus have $y \neq z$ and $F_i z = F_i y$, whence

$$\left|V^{(y,z)} z_i - y_i\right| < \left\|V^{(y,z)} z - y\right\|_\infty \tag{4}$$

On the other hand, $\left|V_i^{(y,z)} z_i - y_i\right| = \left|V_i^{(y,z)} t - s\right| = \left\|V^{(y,z)} z - y\right\|_\infty$ which contradicts Eq. (4). Hence y_{ii}^{x} is injective.

Now let: $C := Q_1 \times \cdots \times Q_{i-1} \times Q_{i+1} \times \cdots \times Q_n$, $J := Q_i$, and $G: C \times J \to R$, $(x, t) \mapsto F_i(x_1, \cdots x_{i-1}, t, x_{i+1}, \cdots, x_n)$. Then G together with $C \times J$, satisfies the hypothesis of Ref.[1], thus proving that $G(x, \cdot) = y_{ii}^{x}$ is on Q_i either strictly isotone for all $x \in Q$ or strictly antitone for all $x \in Q$.

Theorem 7 Let F be Gâteaux-differentiable on Q a convex subset of R^n, and $F'(x)$ be generalized diagonally dominant for all $x \in Q$, then F is quasi-generalized nonlinear diagonally dominant (on Q).

Proof For $i = 1, 2, \cdots, n$, let $x \in Q$, $y \in Q$, $x \neq y$ and $F_i x = F_i y$. By the mean value theorem, there exists $x \in Q$ such that (in general x dependents on both x and y):

$$F_i'(x)(x - y) = \sum_{j=1}^{n} \tfrac{\partial f_i}{\partial x_j}(x)(x_j - y_j) = 0 \tag{5}$$

Because $F'(x)$ be generalized diagonally dominant for all $x \in Q$, then there exist positive numbers $\boldsymbol{u}_i(x)$ such that: $\left|u_i(x) \tfrac{\partial f_i}{\partial x_j}(x)\right| > \sum_{j=1, \neq i}^{n} \left|u_i(x) \tfrac{\partial f_i}{\partial x_j}(x)\right|$. From Eq.(5), we have:

$$\left|u_i(x)^{-1}(x_i - y_i)\right| < \max_{j=1}^{n} \left\{u_j(x)^{-1} \left|x_j - y_j\right|\right\} \tag{6}$$

Construct $V^{(x,y)}: Q \to R^n$ by $V_i^{(x,y)} t = u_i(x)^{-1}(t - x_i) + x_i$ Eq.(6) means $\left|V^{(x,y)} y_i - x_i\right| < \left\|V^{(x,y)} y - x\right\|_\infty$ which finishing our proof.

Remark The following example shows that the convexity of Q in Theorem 7 is necessary. So the conjecture of Ref.[3] is invalid in general.

Example Let $Q = \{(x_1, x_2)^T \mid x_1, x_2 \in R,\ |x_1| \le 3, |x_2| \le 3,\ x_1^2 x_2^2 \le 0.5\}$ define $F: Q \subset R^2 \to R^2$ by $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + \dfrac{x_2^3}{3} \\ x_2 + \dfrac{x_1^3}{3} \end{pmatrix}$, then $F'(x) = \begin{bmatrix} 1 & x_2^2 \\ x_1^2 & 1 \end{bmatrix}$. Obviously $F'(x)$ is generalized diagonally dominant for all $x \in Q$. And $(0,3)^T \in Q, (3,0)^T \in Q$, but $F\left((0,3)^T\right) = (3,3)^T$, $F\left((3,0)^T\right) = (3,3)^T$. Therefore F isn't quasi-generalized nonlinear diagonally dominant thereby is not generalized nonlinear diagonally dominant in sense of Ref.[3].

References

[1] Moré J. Nonlinear generalizations of matrix diagonal dominance with applications to Gauss-Siedel iterations [J]. SIAM J. Numer Anal. , 1972, (9): 357-378

[2]You Zhao-Yong, Wang Chuan-Long. A concept of nonlinear Block Diagonal dominance [J]. Journal Comp Appl Math, 1997, (83): 1-10

[3] Frommer A. Generalized nonlinear diagonal dominance and applications to asynchronous iterative methods [J].Journal Comp Appl Math 1991, (38): 105-124

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4 结 束 语

在模拟试验中，本文采用Java Agent开发框架(Java Agent Development Framework，JADF)作为多Agent系统的开发平台，利用JADF提供的Agent基类和其他功能类，设计并实现了模型的各个Agent。实验表明，MAIHN模型同现有的组网方案相比较，不仅可以实现家庭设备的互联互通，而且具有动态开放、设备协作等特点，适应家庭网络分布、异构和个性化的要求。

参 考 文 献

[1] 蒋建文. 基于多Agent的信息家居系统研究: [博士学位论文][D]. 合肥: 合肥工业大学, 2003

[2] 郭 红, 陈 锐, 胡黎明. 基于XML, CORBA和Agent技术的集成模型研究[J]. 小型微型计算机系统, 2003, 24(9): 1 647-1 649

[3] 代春艳, 张小真. 智能计算机辅助教学系统中Agent的构建[J]. 渝州大学学报, 1998, 15(4): 47-51

[4] 赵怀慈, 黄莎白, 张 霞. 基于Multi-agent的消息通信机制及其实现框架[J]. 系统仿真学报, 2003, 15(11): 1 554-1 557

[5] 范玉顺, 曹军威. 多代理系统理论、方法与应用[M]. 北京:清华大学出版社, 2002. 52-54

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