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Nonlinear Oscillation of Electron on Axis of Charged Ring

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Abstract　The nonlinear oscillation of electron on the central axis of charged ring is studied. After setting up the motion equation, the exact solution and the period of free oscillation are achieved with elliptic integrals. The nonlinearity of oscillation are discussed and illustrated. Truncating expanded series leads to Duffing equation. The approximate expressions of solution and period are gained by Duffing oscillation and harmonic oscillation, which are compared with the exact ones. The appling significance of nonlinear oscillation solution in engineering is indicated.

Key words　charged ring;　nonlinear oscillation;　elliptic integral;　Duffing equation

The theoretic significance and engineering value for oscillation of electron in the electric field are concerned recently. The researches on the nonlinear oscillation in this area are more active lately. The applications become wider and wider[1].

This paper deals with nonlinear oscillation of an electron on the central axis of evenly charged ring. According to the electric field intensity on the axis of the charged ring, the equation of motion for the electron is set up. The exact expressions of solution and period are achieved by elliptic integrating. Their alternations are illustrated and the nonlinear characters are pointed out. Approximated method is to truncate the expanded series which leads to Duffing equation. The solution and period of Duffing oscillation are given. The approximate solutions are compared with the exact ones. The usage area and properties of approximate solutions are investigated. In addition, the applies of nonlinear oscillation solution in engineering are indicated.

1　Equation of Nonlinear Oscillation

In Fig. 1, a ring-shaped conductor of radius R evenly carries a total charge Q. X-axis goes from the center and perpendicular to the plane of the ring. An electron with mass m and charge $-e$ is on the axis. Its motion is investigated in the following.

The electric field on the axis produced by the evenly charged ring is[2]

$$E = \frac{Qx}{4\pi\varepsilon_0 (R^2 + x^2)^{\frac{3}{2}}} \tag{1}$$

Then the motion equation of the electron is expressed as

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{Qex}{4\pi\varepsilon_0 (R^2 + x^2)^{\frac{3}{2}}} = 0 \tag{2}$$

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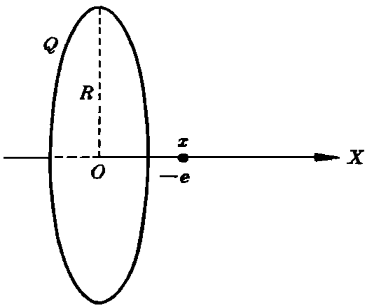

Fig. 1 Oscillation of electron on the axis of evenly charged ring

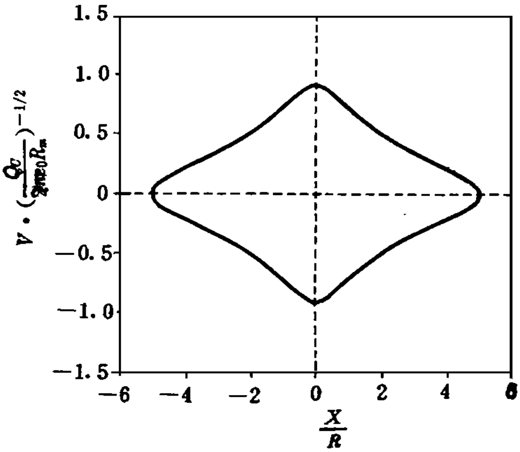

Fig. 2 A phase orbit($x_0=5R, v_0=0$)

The potential energy of the system is

$$W_p = \frac{Qe}{4\pi\varepsilon_0(R^2+x^2)^{\frac{1}{2}}} \tag{3}$$

When $Q>0$, W_p gets its minimum on the ring center $x=0$. There balance is stable. In phase plane, the center is $(0, 0)$.

The energy integral of Eq. (2) is

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - \frac{Qe}{4\pi\varepsilon_0\sqrt{R^2+x^2}} = H_0 \tag{4}$$

where H_0 is the Hamiltonian of the system.

Noting velocity as $v=\frac{dx}{dt}$, Eq. (4) gives the closed phase orbits around (0, 0) in phase plane, which indicates the periodic oscillation of the electron about the positively charged ring center on the axis. The oscillation is nonlinear. A phase orbit is shown in Fig. 2

2 Exact Solution and Period

Suppose the initial conditions for the oscillation Eq. (2) are

$$x|_{t=0} = x_0 \geqslant 0; \quad v|_{t=0} \tag{5}$$

and note the angular frequency of correspondent harmonic oscillation as ω_0, hence

$$\omega_0^2 = \frac{Qe}{4\pi\varepsilon_0 mR^3} \tag{6}$$

Evidently, Eq. (4) has a asymmetry to X-axis and V-axis in the phase plane. First we slove it in the fourth quadrant. The solutions in other quadrants can be obtained from the symmetry. From Eq. (4) we get

$$t = \int_{x_0}^{x}\frac{dx}{v} = \frac{1}{\sqrt{2}\,\omega_0 R}\int_{x}^{x_0}\frac{dx}{\sqrt{\frac{R}{\sqrt{R^2+x^2}} - \frac{R}{\sqrt{R^2+x_0^2}}}} \tag{7}$$

Introducing quantities without dimension

$$y_0 = \frac{R}{\sqrt{R^2 + x_0^2}} \qquad y_0 \in (0, 1] \tag{8}$$

$$y = \frac{R}{\sqrt{R^2 + x^2}} \qquad y \in [y_0, 1] \tag{9}$$

Change Eq. (4) into[3]

$$t = \frac{1}{\sqrt{2}\,\omega_0}\int_{y_0}^{y} \frac{\mathrm{d}y}{y^2\sqrt{-(y-1)(y-y_0)(y+1)}} =$$

$$\frac{1}{2\sqrt{2}\,\omega_0 y_0}\left[\frac{2\sqrt{-(y-1)(y-y_0)(y+1)}}{y} + \int_{y_0}^{y}\frac{\mathrm{d}y}{y\sqrt{-(y-1)(y-y_0)(y+1)}} + \int_{y_0}^{y}\frac{y\,\mathrm{d}y}{\sqrt{-(y-1)(y-y_0)(y+1)}}\right] \tag{10}$$

Using elliptic integration yields

$$t = \frac{1}{\omega_0 y_0}\left[E(\varphi,k) - F(\varphi,k) + \frac{y_0+1}{2y_0}\Pi\left(\frac{1-y_0}{2y_0},\varphi,k\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{y} - \frac{1}{(y_0+1)(y+1)}\right)\sqrt{-(y-1)(y-y_0)(y+1)}\right] \tag{11}$$

where $F(\varphi,k)$, $E(\varphi,k)$, $\Pi(n,\varphi,k)$ are the first kind, second kind and third kind of Legender normal elliptic integrals, respectively. The module k is

$$k = \sqrt{\frac{1-y_0}{2}} \tag{12}$$

Angle φ satisfies

$$\varphi = \arcsin\sqrt{\frac{2(y-y_0)}{(1+y)(1-y_0)}} \qquad \varphi \in [0, \frac{\pi}{2}] \tag{13}$$

The function curve of Eq. (11) is plotted in Fig. 3 with solid line. The dotted line is a synchronous cosine curve of reference.

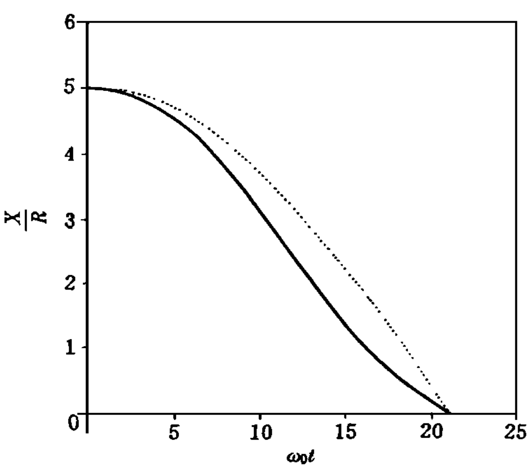

Fig. 3 Oscillation curve during one fourth period ($x_0 = 5R$)

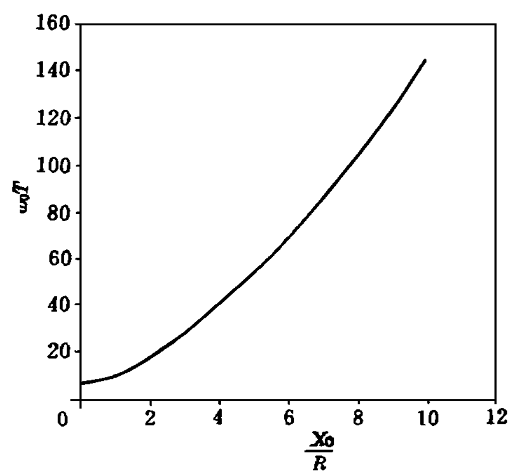

Fig. 4 Function curve of period versus amplitude

Employing the symmetry of phase orbits and Eq. (10), we achieve the period of oscillation

$$T = 4\int_0^{x_0}\frac{dx}{v} = \frac{2}{k_0(1-k^2)}\left[2E(k) - K(k) + C(-2k^2, k)\right] \qquad (14)$$

where $K(k)$, $E(k)$, $C(d, k)$ are the first, second and third kind of Legender complete elliptic integrals, respectively. The module k is the same as that in Eq. (2). Eq. (14) demonstrates that the period is the function of amplitude, which clearly exhibits the nonlinear characters in the oscillation. The function curve is shown in Fig. 4.

If expand $E(k)$, $K(k)$, $C(-2k^2, k)$ into series respectively and keep the terms to k^2, we have

$$T = \frac{\pi}{4k_0}\left(9\frac{\sqrt{R^2 + x_0^2}}{R} - 1\right) \qquad (15)$$

Keep the first term only, Eq. (15) can be approximated as

$$T_0 = \frac{2\pi}{k_0} \qquad (16)$$

3 Approximate Solution and Period

Now we intend approximate solution of oscillation with small amplitude around the ring center. Expanding the second term on the left side of Eq. (4) into Mechlolin series and neglecting $O\left(\frac{x^6}{R^6}\right)$ yields

$$\left(\frac{dx}{dt}\right)^2 + k_0^2x^2 - \frac{3k_0^2}{4R^2}x^4 = H = H_0 + \frac{Qe}{4\pi X_0 R} \qquad (17)$$

that is corresponding to Duffing equation[4]. The oscillation solution is expressed by

$$x = f\,\mathrm{sn}\left(\frac{K(\kappa)(2kt + \pi)}{\pi}, \kappa\right) \qquad (18)$$

where $\mathrm{sn}(h, \kappa)$, is Jacobi elliptic function; κ is the module; k is the angular frequency; κ, f and k satisfy

$$\frac{\kappa}{\kappa^2 + 1} = \frac{\sqrt{3H}}{2k_0R} \qquad (19)$$

$$f = \frac{\kappa}{2R}\sqrt{\frac{6}{(\kappa^2 + 1)}} \qquad (20)$$

$$k = \frac{\pi k_0}{2K(\kappa)\sqrt{\kappa^2 + 1}} \qquad (21)$$

Expanding Eq. (18) into Fourier seies, we get

$$x = f\frac{\pi}{\kappa K(\kappa)}\sum_{m=1}^{\infty}\mathrm{csch}\left[\left(m - \frac{1}{2}\right)\frac{K'(\kappa)}{K(\kappa)}\pi\right]\cos\left[(2m - 1)kt\right] \qquad (22)$$

where

$$K'(\kappa) = K(\kappa') \qquad (23)$$

κ' is the complementary module of κ. The terms of high frequency in Eq. (22) decline quickly. Keep the first term only there results

$$x = f\frac{\pi}{\kappa K(\kappa)}\mathrm{csch}\left[\frac{K'(\kappa)}{2K(\kappa)}\pi\right]\cos kt \qquad (24)$$

Furthermore, expanding Eq. (21) in modules κ and keeping to the term of $\left(\frac{x_0}{R}\right)^2$ leads to

$$T = \frac{2\pi}{k_0}\left[1 + \frac{9}{16}\left(\frac{x_0}{R}\right)^2\right] \qquad (25)$$

It is not difficult to prove that the order of $(\frac{x_0}{R})^2$ Eq. (25) and Eq. (15) are equivalent.

As a comparison, the exact expression, Duffing oscillation expression and harmonic oscillation expression for solution and period are plotted together in Fig. 5 and Fig. 6. There, solid line is for the exact expression, dashed line for Duffing oscillation and dotted line for harmonic oscillation. We can see Duffing oscillation and harmonic oscillation have approximate values only when the amplitued is small. For these two kinds of approximated solutions under small amplitued, besides the numerical value are more closed to the exact solution, Duffing oscillation represents the basic property of alternation of period versus amplitued, which shows that Duffing oscillation is better than the harmonic one.

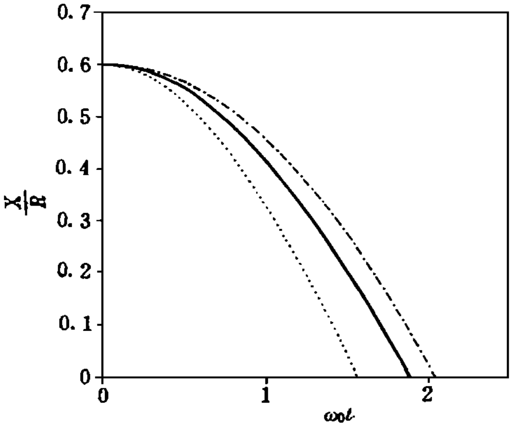

Fig. 5 Oscillation curves for three solutions (x_0= 0.6R)

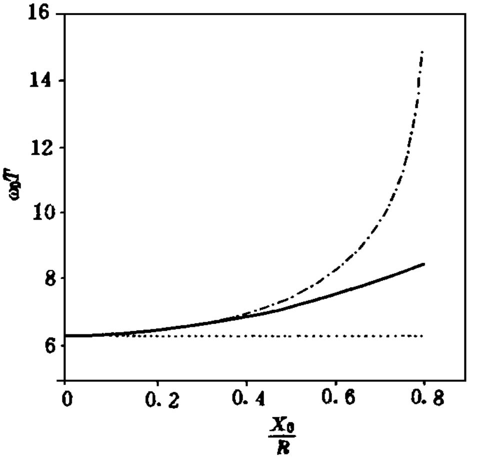

Fig. 6 Period curves for three solutions

4 Conclusion

The motion of electron on the axis of positively charged ring is nonlinear oscillation. The exact expressions of solution and period for free oscillation are calculated by elliptic integration. The period is the function of amplitude. In the case of small amplitued, the approximate forms are Duffing oscillation and harmonic oscillation. Both approximate solutions and periods can be achieved by trucating expanded series for exact expression to the proper term. The nonlinear oscillation solution could be applied in engineering to remedy the defect of linear oscillation solution under large amplitude, such as correctly designing the frequency-amplitude curve, exactly calculating the electron transit angle in vacuum tube, effectively modulate the mass spectrometer, deeply analyzing the chromatism in electron optics etc.

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电子在电环轴上的非线性振动

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【摘要】 研究电子在电环轴上的非线性振动,在建立电子的运动方程后利用椭圆积分求得自由振动的准确解及周期,讨论和图示其非线性振动特征。截断级数展式导致杜芬方程,解和周期近似表达式由杜芬振动和简谐振动给出,且与准确表达式比较。指出了非线性振动解在工程上的实用意义。

关　键　词　带电环; 非线性振动; 椭圆积分; 杜芬方程
中图分类号　O422

编辑　叶　红

· 科研成果介绍 ·

NH4450型 100 MHz智能数字存储示波器

主研人员: 郭戌生　陈长龄　王树菁　童　劲　彭大军

该示波器采用随机取样技术可对被测信号进行取样量化、存储、处理参数、自动测量和显示,因此该仪器能用于检测军用电子元器件,设备稳态和瞬态特性,借助传感技术,还可用于各种非电事件的分析与评估,广泛应用于机械、电力、军用武器、飞行器和火箭核爆炸等各个领域的仪器等。其主要技术指标: 等效带宽 100 MHz,垂直精度 ≤ 3%,相对精度≤ 1%,数字化速率 10 MSPS,幅度分辨率 8 bit,时间分辨率 100 PS,参数自动测量。

该仪器性能相当于 80年代 HP公司 54501A型数字存储示波器水平。

· 科　卞 ·

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