第27卷 第2期 电子科技大学学报 Vol.27 No.2
1998年4月 Journal of UEST of China Apr.1998

Major-in-weight Efficiency and Major-in-weight Optimality of Multiple Criteria Decision Making *

Chen Zhanfeng
(Department of Statistics, Shanghai University of Finance and Economics Shanghai 200433)
Xu Duan
(Dept. of Statistics, Hebei University of Economics and Trade Shijiazhuang 050061)
Wu Jianzhong
(Institue of Systems Engineering, Shanghai Jiao Tong University Shanghai 200030)

Abstrace A new efficiency theory of multiple criteria decision making is established in this paper. The new effciency is based on the differences among weights of the criteria, with "the larger the weight sum of the better criteria is, the better the related alternative" being the preference principle. The key definitions in this paper are β— major-in-weight efficient solution and β— major-in-weight optimal solution. Some properties of these solutions are shown. The relationships between them and Pareto efficient (optimal) solution are discussed.

Key words multiple criteria decision making; generalized weighting vector; β— major-in-weight efficient solution; β— major-in-weight optimal solution

The notion of "solution" is very important in multiple criteria decision making. Based on diferent decision rules, different kinds of solutions depict different aspects of decision making. There already exist many efficient solutions, such as Pareto efficient solutions[1], proper efficient solutions[2, 3] and major efficient solutions[4]. A common implication in these kinds of efficiency is that every criterion has the same weight.

Nevertheless, in multiple criteria decision making, the decision maker often treats different criteria with different weights. There have been weighted-norm approach[1] and weighted-sum approach[5] to deal with such case. This paper, from the viewpoint of "noninferiority", in efficiency sense, makes an intensive study on this topic.

1 β—Major-in-weight Efficient (Optimal) Solution

Consider the following multiple criteria decision problem (MCDP):

$$\max f(x) \quad (1)$$
$$\text{s.t. } x \in X$$

where $X \subset R^n$ is the feasible region in dexcision space, $f: X \to R^m$, $f(x) = \{f_1(x), f_2(x), \cdots, f_m(x)\}$: $f_i: X \to R$ ($i = 1, 2, \cdots, m$) are real-valued functions; R^n and R^m are Euclidean spaces.

Received 14, 1997 November
* Supported by the Doctor Advisors' Funds of Shanghai Development Research Institute, Development Research Center of the State Council of China (No. 960005)

We intoduce two cone sets in R^m, C_β and $\overset{\circ}{C}_\beta$:

$$C_\beta = \{ v \mid v = (v_1, v_2, \cdots, v_m)^T \in R^m, \sum_{i=1}^{m} \beta_i \mathrm{sgn}(v_i) \geqslant 0 \}$$

$$\overset{\circ}{C}_\beta = \{ v \mid v = (v_1, v_2, \cdots, v_m)^T \in R^m, \sum_{i=1}^{m} \beta_i \mathrm{sgn}(v_i) > 0 \}$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_m)^T$ satisfies $0 \leqslant \beta_i \leqslant 1$ ($i = 1, 2, \cdots, m$) and $\sum_{i=1}^{m} \beta_i = 1$. sgn(°) is the Keroneker function

$$\mathrm{sgn}(t) = \begin{cases} 1 & \text{when } t > 0 \\ 0 & \text{when } t = 0 \\ -1 & \text{when } t < 0 \end{cases}$$

re

In multiple criteria decision making, β_i is in fact a kind of generalized weight. The value of β_i is relative to w_i, the weight of the ith criterion, and $|\Delta y_i|$ s, the absolute differences of the ith objective values of two different alternatives. β is called generalized weighting vector.

Using C_β and $\overset{\circ}{C}_\beta$, we define cone ordering relations $\geqslant_{C_\beta}$ and $>_{C_\beta}$ as follows:

$$a \geqslant_{C_\beta} b \Leftrightarrow a - b \in C_\beta \quad \text{for any } a, b \in R^m$$

$$a >_{C_\beta} b \Leftrightarrow a - b \in \overset{\circ}{C}_\beta \quad \text{for any } a, b \in R^m$$

The two ordering relations defined by the above are called $\beta-$major-in-weight orders. They possess the following properties.

Proposition 1

1) $\geqslant_{C_\beta}$ is reflective, not antisymmetric, not transitive, and complete.

2) $>_{C_\beta}$ is not reflective, not antisymmetric, not transitive, and not complete.

3) There must be one and only one holds, either $a \geqslant_{C_\beta} b$ or $b >_{C_\beta} a$, for any $a, b \in R^m$.

Definition 1 Let $X \subset R^n$, $f: X \to R^m$. Then,

1) $\bar{x} \in X$ is $\beta-$major-in-weight efficient solution to Eq. (1) iif there does not exist $x \in X$ such that $f(x) \geqslant_{C_\beta} f(\bar{x})$ and $f(x) \neq f(\bar{x})$. Let $E_{C_\beta}(X)$ denote the set of all $\beta-$major-in-weight efficient solutions.

2) $x^* \in X$ is $\beta-$major-in-weight optimal solutions to Eq. (1) iif $f(x^*) \geqslant_{C_\beta} f(x)$ holds for all $x \in X$. Let $O_{C_\beta}(X)$ denote the set of all $\beta-$major-in-weight optimal solutions.

According to the definition of cone C_β, $\bar{x}$ is $\beta-$major-in-weight efficient solution means that $\bar{x} \in X$ and there does not exist $x \in X$ such that $\sum_{i=1}^{m} \beta_i \mathrm{sgn}(f_i(x) - f_i(\bar{x})) \geqslant 0$ and $f(x) \neq f(\bar{x})$. x^* is $\beta-$major-in-weight optimal solution means that $x^* \in X$ and $\sum_{i=1}^{m} \beta_i \mathrm{sgn}(f_i(x^*) - f_i(x)) \geqslant 0$ for all $x \in X$.

2 Properties of $\beta-$Major-in-weight Efficient (Optimal) Solutions

Theorem 1 Let $X \subset R^n$, $f: X \to R^m$. Then:

1) $\bar{x} \in E_{C_\beta}(X)$ iif $\bar{x} \in X$, and $f(\bar{x}) >_{C_\beta} f(x)$ holds for all $x \in X$ and $f(x) \neq f(\bar{x})$.

2) $x^* \in O_{C_\beta}(X)$ iif $x^* \in X$, and there does not exist $x \in X$ such that $f(x) \succ_{C_\beta} f(x^*)$.

Proof

1) $\bar{x} \in E_{C_\beta}(X)$ means that $\bar{x} \in X$, and there does not exist $x \in X$ such that $f(x) \geqslant_{C_\beta} (\bar{x})$ and $f(x) \neq f(\bar{x})$. By Proposition 1.3), it implies that $f(\bar{x}) \succ_{C_\beta} f(x)$ must hold for any $x \in X$ and $f(x) \neq f(\bar{x})$.

2) $x^* \in O_{C_\beta}(X)$ means that $x^* \in X$, and $f(x^*) \geqslant_{C_\beta} f(x)$ holds for all $x \in X$. By Propostion 1.3), it implies that there does not exist $x \in X$ such that $f(x) \succ_{C_\beta} f(x^*)$.

Theorem 2　Let $X \subset R^n$, $f: X \to R^m$. Then $\bar{x} \in E_{C_\beta}(X) \Rightarrow f(E_{C_\beta}(X)) = \{f(\bar{x})\}$

Proof　Suppose $y \in f(E_{C_\beta}(X))$ but $y' \neq f(\bar{x})$. Let $y' = f(x')$, $x' \in E_{C_\beta}(X)$. Compare $f(\bar{x})$ and $f(x')$. If $f(x') \geqslant_{C_\beta} f(\bar{x})$, then there exists $x' \in X$ such that $f(x') \geqslant_{C_\beta} f(\bar{x})$ and $f(x') \neq f(\bar{x})$. This contradicts $\bar{x} \in E_{C_\beta}(X)$. Otherwise, $f(\bar{x}) \succ_{C_\beta} f(x')$ holds, we get contradiction with $x' \in E_{C_\beta}(X)$. Hence, $f(E_{C_\beta}(X)) = \{f(\bar{x})\}$ holds.

Theorem 2 demonstrates that if there does exist major-in-weight efficient solution to Eq. (1), the image set of $E_{C_\beta}(X)$ is single-point set.

Theorem 3　Let $X \subset R^n$ be convex. $f: X \to R^m$ is a concave vector function, and there is at least one component function which is strictly concave. The $\bar{x} \in E_{C_\beta}(X) \Rightarrow E_{C_\beta}(X) = \{\bar{x}\}$.

Proof　Suppose $x' \in E_{C_\beta}(X)$ but $x' \neq \bar{x}$. From Theroem 2, we have $f(x') = f(\bar{x})$. Since X is convex, $x = \frac{x'}{2} + \frac{\bar{x}}{2} \in X$. By the concavity of f, we have

$$f(x) \geqslant \frac{f(x')}{2} + \frac{f(\bar{x})}{2} = f(\bar{x})$$

and there exists at least one t such that

$$f_t(x) \geqslant \frac{f_t(x')}{2} + \frac{f_t(\bar{x})}{2} = f_t(\bar{x})$$

Thus, we can deduce that $f(x) \neq f(\bar{x})$ and $\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x) - f_i(\bar{x})) > 0$ hold.

The above implies that there exists $x \in X$ such that $f(x) \geqslant_{C_\beta} f(\bar{x})$ and $f(x) \neq f(\bar{x})$. Contradicts $\bar{x} \in E_{C_\beta}(X)$.

Theorem 4　Let $X \subset R^n$, $f: X \to R^m$. Then $E_{C_\beta}(X) \neq \emptyset \Rightarrow E_{C_\beta}(X) = O_{C_\beta}(X)$.

Proof　$E_{C_\beta}(X) \subseteq O_{C_\beta}(X)$. Let $\bar{x} \in E_{C_\beta}(X)$. Then, $\bar{x} \in X$ and there does not exist $x \in X$ such that $f(x) \geqslant_{C_\beta} f(\bar{x})$ and $f(x) \neq f(\bar{x})$. This implies that either $f(\bar{x}) \succ_{C_\beta} f(x)$ or $f(\bar{x}) = f(x)$ holds for all $x \in X$. If $f(\bar{x}) = f(x)$ holds, $f(\bar{x}) \geqslant_{C_\beta} f(x)$ holds as well. Hence $f(\bar{x}) \geqslant_{C_\beta} f(x)$ holds for all $x \in X$, the definition of $\bar{x} \in O_{C_\beta}(X)$. Thus, we have $E_{C_\beta}(X) \subseteq O_{C_\beta}(X)$.

$E_{C_\beta}(X) \supseteq O_{C_\beta}(X)$. Let $x^* \in O_{C_\beta}(X)$. Then, $f(x^*) \geqslant_{C_\beta} f(x)$ holds for all $x \in X$. Suppose $x^* \notin E_{C_\beta}(X)$. Since $E_{C_\beta}(X) \neq \emptyset$, there exists an $x' \in E_{C_\beta}(X)$, $x' \neq x^*$. Thus $f(x^*) \geqslant_{C_\beta} f(x')$ holds. If $f(x^*) \neq f(x')$, we arrive at the contradiction with $x' \in E_{C_\beta}(X)$. If $f(x^*) = f(x')$, by Theorem 2, we have $x^* \in E_{C_\beta}(X)$, which is a contradiction with the supposition. Hence $x^* \in E_{C_\beta}(X)$, followed by $E_{C_\beta}(X) \supseteq O_{C_\beta}(X)$.

Substitute the cone C_β and $\mathring{C}_\beta$ with R_+^m and $\mathring{R}_+^m$ respectively

$$R_+^m = \{ v \mid v = (v_1, v_2, \cdots, v_m)^T \in R^m, v_i \geqslant 0\}$$

$$\mathring{R}_+^m = \{ v \mid v = (v_1, v_2, \cdots, v_m)^T \in R^m, v_i > 0\}$$

Similar to Definition 1, we can obtain the definitions of Pareto efficient solution and Pareto optimal solution. Let $E(X)$ and $O(X)$ denote the set of all Pareto efficient solutions and the set of all Pareto optimal solutions respectively.

Theorem 5 Let $X \subset R^n$, $f: X \to R^m$. Then $O(X) \subseteq E_{C_\beta}(X) \subseteq O_{C_\beta}(X) \subseteq E(X)$.

Proof $O(X) \subseteq E_{C_\beta}(X)$. Let $x^* \in O(X)$. Then, $x^* \in X$, and $f(x^*) \geqslant f(x)$ holds for all $x \in X$. If $f(x^*) \neq f(x)$, there must exist one t such that $f_t(x^*) > f_t(x)$, hence

$$\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x^*) - f_i(x)) > 0$$

This means that

$$\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x) - f_i(x^*)) < 0$$

holds for any $x \in X$ and $f(x^*) \neq f(x)$. Thus, there does not exist $x \in X$ such that

$$\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x) - f_i(x^*)) \geqslant 0$$

and

$$f(x) \neq f(x^*)$$

The definition of $x^* \in E_{C_\beta}(X)$.

$E_{C_\beta}(X) \subseteq O_{C_\beta}(X)$ (See Theorem 4). $O_{C_\beta}(X) \subseteq E(X)$. Suppose $x^* \in O_{C_\beta}(X)$. Then, $x^* \in X$, and

$$\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x^*) - f_i(x)) \geqslant 0$$

for all $x \in X$. This implies that

$$\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x) - f_i(x^*)) \leqslant 0$$

for all $x \in X$. Thus, we can deduce that there does not exist $x \in X$ such that $f(x) \geqslant f(x^*)$ and $f(x) \neq f(x^*)$ (otherwise, we may get $\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x) - f_i(x^*)) > 0$, i.e. $\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x^*) - f_i(x)) < 0$. Then, we arrive at a contradiction with $\sum_{i=1}^{m} \beta_i \operatorname{sgn}(f_i(x^*) - f_i(x)) \geqslant 0$).

That is the definition of $x^* \in E(X)$.

Theorem 6 Let $X \subset R^n$, $f: X \to R^m$. If $O(X) \neq \varnothing$, then, $O(X) = E_{C_\beta}(X) = O_{C_\beta}(X) = E(X)$.

Proof By the property of Pareto efficient solution, if $O(X) \neq \varnothing$, then

$$O(X) = E(X)$$

From Theorem 5, we have $O(X) = E_{C_\beta}(X) = O_{C_\beta}(X) = E(X)$.

3 Conclusions

This paper advances a new effieiency theory for multiple criteria decision making. In addition to what we have done, many other topics in this area yet needs to be studied, such as the effective method for finding major-in-weight efficient (optimal) solutions. For linear multiple criteria decision problems, we may obtain some special results and methods for major-in-weight efficiency (optimality). All these remain for further research in our future papers.

References

1 Changkong V, Haimes Y Y. Multiobjective decision making: theory and methodology. New York: North-Holland, 1993

2 Geoffrin A M. Proper efficiency and the theory of vector maximization. Journal of Mathematical Analysis and Applications, 1968, (22): 618~630

3 Benson H P, Morin T L. The vector maximization problem: proper efficiency and stability. SLAM Journal on Applied Mathematics. 1977, (32): 64~72

4 Hu Y D. Major optimality and major efficiency in multicriteria optimaization. Optimization Techniques and Applications, World Scientific, 1992, (1): 368~374

5 Steuer R E. Multiple criteria optimization: theory, computation, and application. New York: John Wiley & Sons Inc, 1996

6 Yu P L. Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiple objectives. Journal of Optimization Theory and Applications, 1974, (14): 319~377

多目标决策的较重有效性与较重最优性 *

陈占锋 * *　　许　端

（上海财经大学统计系　上海　200433）　（河北经贸大学统计系　石家庄　050061）

吴健中

（上海交通大学系统工程研究所　上海　200052）

【摘要】 基于不同目标的权重不同，以“好目标的权重和越大，相应的方案越好”作为决策准则，提出了一种新的多目标决策有效性理论。其关键概念是β—较重有效解与β—较重最优解。文中证明了解的性质。讨论了该类解与 Pareto—有效解的关系。

关　键　词　多目标决策；　广义权重向量；　β—较重有效解；　β—较重最优解

中图分类号　O221

编辑　叶　红

1997 年 11 月 14 日收稿

* 国务院发展研究中心上海发展研究所博士导师专项科研基金资助（编号：960005）

* * 男　32 岁　博士　讲师。现在上海财经大学博士后流动站工作