On Negative Solution of Invariant Subspace Problem**

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Abstract　In this paper, the negative solution of the invariant subspace problem is simplified, and a bounded linear operator on l_1 without non-trivial closed invariant subspaces is presented. In particular, the estimations of norms of some vectors are replaced by the calculation of coordinates of the vectors in this paper and the simple algbraice operations are employed instead of the technique of compact spaces.

Key Words　Banach space;　bounded linear operator;　invariant subspace;　algebra;　norm

Recently, P. Enflo and C. J. Read presented the counterexample for the problem of invariant subspaces on nonreflexive Banach spaces respectively[1,2]. B. Beauzamy, C. J. Read and A. M. Davie simplified the result of Enflo and Read respectively[3~5]. In this paper, we furtherly simplify the result by means of algebraic techniques.

1　The Operator T on l_1 with Required properties

Let N denote the set of all positive integers and Z^+ denote all nonnegative integers. Let $d=\{d_n\}_{n=0}^{\infty}$ denote a strictly increasing sequence of positive integers, which is required to increase sufficiently rapidly in Ref. [6], and $d_0=1$. We shall write $a_n=d_{2n-1}$, $b_n=d_{2n}$ for all n in N. So $1<a_1<b_1<a_2<b_2<\cdots<a_n<b_n<\cdots$. We shall also define $v_0=0$, $v_n=(n-1)(a_n+b_n)$ for each n in N.

Let $\|x\|$ denote the ordinary norm of the vector x in the space l_1. Let $\{f_n\}_{n=0}^{\infty}$ denote the unit vector basis of l_1, where $f_0=(1, 0, 0, \cdots, 0, \cdots)$, $f_1=(0, 1, 0, \cdots, 0, \cdots)\cdots$. Let E_n denote the subspace of l_1 spanned by the set $\{f_i; 0\leqslant i\leqslant n\}$. Let E denote the dense subspace of l_1 spaned by the set $\{f_i; i\in Z^+\}$.

Lemma 1　Let $d=\{d_n\}_{n=0}^{\infty}$ is a strictly increasing sequence of positive integers, provided d increases sufficiently rapidly (abbreviation p. d.), then there is a unique sequence $\{e_i\}_{i=0}^{\infty}$ in E such that $\{e_i; i\in Z^+\}$ is a linearly independent set in E and satisfies the follwing conditions:

1) $f_0=e_0$;

2) If integers n, r, i satisfy $n\geqslant 2$, $1\leqslant r\leqslant n-1$, and $ra_n\leqslant i\leqslant ra_n+v_{n-r-1}$, then $f_i=a_{n-r}(e_i-e_{i-a_n})$.

3) If integers n, r, i satisfy $n\geqslant 2$, $1\leqslant r\leqslant n-1$, and $(r-1)a_n+v_{n-r}<i<ra_n$, then $f_i=2^{[(r-1/2)a_n-i]/\sqrt{a_n}}e_i$.

4) If integers n, r, i satisfy $n\geqslant 2$, $1\leqslant r\leqslant n-1$, and $r(a_n+b_n)\leqslant i\leqslant(n-1)a_n+rb_n$, then $f_i=e_i$

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$-b_n e_{i-b_n}$.

5) If integers n, r, i satisfy $n\geqslant 2$, $1\leqslant r\leqslant n-1$, and $(n-1)a_n+(r-1)b_n<i<r(a_n+b_n)$, then $f_i=2^{[(r-1/2)b_n-i]/\sqrt{b_n}}e_i$.

6) span $\{e_i; 0\leqslant i\leqslant n\}=E_n$ for all n in Z^+.

7) span$\{e_i; i\in Z^+\}=E=$span$\{f_i; i\in Z^+\}$.

8) If integers n, r, i satisfy $n\geqslant 2$, $1\leqslant r\leqslant n-1$, and $ra_n\leqslant i\leqslant ra_n+v_{n-r-1}$, then $\|e_i-e_{i-ra_n}\|\leqslant 2/a_{n-r}$.

9) If integers n, r, i satisfy $n\geqslant 2$, $1\leqslant r\leqslant n-1$, and $r(a_n+b_n)\leqslant i\leqslant (n-1)a_n+rb_n$, then $\|e_i-e_{i-rb_n}\|\leqslant 2b_n^{r-1}$.

The verification of Lemma 1 is essentially the same as the argument of (5.0.1) in Ref.[6], note 2.2 in Ref.[4] and formulas (1), (2) in Ref.[5], and is therefore omitted.

Definition 1　Let $T: E\rightarrow E$ be the linear operator such that $T(e_i)=e_{i+1}$, $i\in Z^+$.

Theorem 1　Provided d increases sufficiently rapidly, T is a bounded linear operator on $(E, \|\circ\|)$ and $\|T\|\leqslant 2$.

The proof of Theorem 1 proceeds in the same way as that of Lemma 4.1 in Ref.[4] or Lemma 2 in Ref.[5], and is therefore omitted.

Note 1　For the sake of convenience, we shall just write T in Theorem 1 for its extension to all of l_1.

2　An Auxiliary Operator and An Auxiliary Norm

Definition 2　Let $Q_m(m>2))$ be the linear operator from E into $E_{(m-1)a_m}$ such that

$$Q_m f_i=\begin{cases} f_i & 0\leqslant i\leqslant (m-1)a_m \\ -a_m e_j & i=(n-m)a_n+j,\ n>m, 0\leqslant j\leqslant v_{m-1} \\ 0\ ,0 & \text{otherwise} \end{cases}$$

We observe that if $i=(n-m)a_n+j$, $n>m$, and $0\leqslant j\leqslant v_{m-1}$, then $i\geqslant (n-m)a_n>a_n>(m-1)a_m$.p.d.Consequently, Definition 2 is well defined.

Lemma 2　$Q_m(m>2)$ is a bounded linear operator from $(E, \|\circ\|)$ into $(E_{(m-1)a_m}, \|\circ\|)$, and

there is a constant C_m depending only on the numbers a_1, b_1, $\cdots$, a_{m-1}, b_{m-1}, a_m such that $\|Q_m x\|\leqslant C_m\|x\|$ for all x in E.p.d.

Proof　Using Lemma 1 and the algebraic operation of vectors, we can prove Lemma 2.

Note 2　We may assume without loss of generality that Q_m is a bounded linear operator from l_1 into $(E_{(m-1)a_m}, \|\circ\|)$ and $\|Q_m\|\leqslant C_m$.p.d.

Lemma 3　If $m>2$, and $a_m+b_m\leqslant s\leqslant (m-1)a_m+b_m$, then $T^s(I-Q_m)$ is a bounded linear operator on l_1 and $\|T^s(I-Q_m)\|\leqslant 2$, p.d.where I denote the identity operator.

The proof of Lemma 3 proceeds in the same way as that of Lemma 4.1 in Ref.[4] or Corollary of Lemma 3 in Ref.[5], and is therefore omitted.

Definition 3　If $y=y_0e_0+y_1e_1+\cdots+y_ne_n$, $n\in Z^+$, and $y\neq 0$, then we write val$(y)=\min\{i; y_i$

$\neq 0\}$.

Lemma 4 Let $k\geqslant 3$ be an integer. If $x\in l_1$ and $x\neq 0$, then there exist m_0, r_0 in N such that $m_0\geqslant r_0+k$, $Q_{m_0}x\neq 0$, and $\mathrm{val}(Q_{m_0}x)\leqslant r_0 a_{m_0}$.

Proof Suppose the conclusion is not true. Then for any m, $r\in N$ with $m\geqslant r+k$, we have

$$Q_m x=0 \quad \text{or} \quad \mathrm{val}(Q_m x)>ra_m \tag{1}$$

Write $x=\sum_{i=0}^{\infty}x_i f_i$. Then from 6), we can set for any $m>2$

$$\sum_{i=0}^{(m-1)a_m}x_i f_i=\sum_{i=0}^{(m-1)a_m}y_{mi}e_i \tag{2}$$

If $r\in N$, $m\geqslant r+4$ and $ra_m+v_{m-r-2}<i<ra_m+v_{m-r-1}$, we are inside the range of application of fomula 2) for f_i. So we get

$$y_{mi}=x_i a_{m-r} \tag{3}$$

If $m>2$ and $v_{m-1}<i<(m-1)a_m$, we are in the range of application of fomulas 2) and 3) for f_i, and f_i has no term e_i with $v_{m-2}<i<v_{m-1}$. Therefore we can write

$$\sum_{i=v_{m-2}+1}^{v_{m-1}}x_i f_i-\sum_{i=v_{m-2}+1}^{v_{m-1}}y_{mi}e_i=\sum_{i=0}^{v_{m-2}}w_i f_i \tag{4}$$

By Note 2, Q_m is a continuous linear operator from l_1 into $(E_{(m-1)a_m}, \|\circ\|)$. Thus by the definition of Q_m (m>2), we can also write $Q_m(\sum_{i=(m-1)a_m+1}^{\infty}x_i f_i)=\sum_{i=0}^{v_{m-1}}z_{mi}e_i$, where

$$z_{mi}=-a_m\sum_{n=m+1}^{\infty}x_{(n-m)a_n+i} \tag{5}$$

Thus by Eq. (2) and Definition 2 we have $Q_m x=\sum_{i=0}^{(m-1)a_m}y_{mi}e_i+\sum_{i=0}^{v_{m-1}}z_{mi}e_i$ for each $m>2$. Therefore it follows from Eq. (1) that if $r\in N$, $m\geqslant r+k+1$ and $v_{m-1}<i\leqslant(r+1)a_m$, then

$$y_{mi}=0 \tag{6}$$

and if $r\in N$, $m\geqslant r+k+1$ and $0\leqslant i\leqslant v_{m-1}$, then $y_{mi}+z_{mi}=0$. In particuar, if $m\geqslant k+2$, $0\leqslant i\leqslant v_{m-1}$, then

$$y_{mi}+z_{mi}=0 \tag{7}$$

If $r\in N$, $m\geqslant r+k+1$ and $ra_m+v_{m-r-2}<i\leqslant ra_m+v_{m-r-1}$, then, noting that $v_{m-1}<ra_m+v_{m-r-2}$, p. d. $ra_m+v_{m-r-1}<(r+1)a_m$, p. d. and using Eq. (4) and Eq. (6), we get

$$x_i=(1/a_{m-r})y_{mi}=0 \tag{8}$$

For any n, m in N with $n>m$, set $r=n-m$. Hence if $m\geqslant k+2$ and $v_{m-2}<i\leqslant v_{m-1}$, then $n\geqslant r+k+1$ and $ra_n+v_{n-r-2}<(n-m)a_n+i\leqslant ra_n+v_{n-r-1}$, p. d. Therefore it follows from Eq. (5) and Eq. (8) that $z_{mi}=0$. Thus by Eq. (7) we get $y_{mi}=0$ whenever $m\geqslant k+2$ and $v_{m-2}<i<v_{m-1}$. Hence it follows from Eq. (4) that if $m\geqslant k+2$ and $v_{m-2}<i<v_{m-1}$, then $x_i=0$. Consequently, if $n\geqslant k+1$, $v_{n-1}<i<v_n$, then

$$x_i=0 \tag{9}$$

Hence, noting that $(m-1)a_m<v_m$, p. d., we have $x_i=0$ whenever $m\geqslant k+1$ and $v_{m-1}<i\leqslant(m-1)a_m$. Therefore it follows from 1) ~ 5) and Eq. (2) that $y_{mi}=0$ whenever $m\geqslant k+1$ and $v_{m-1}<i\leqslant(m-1)a_m$. Consequently, using Eq. (2) for $m\geqslant k+1$, we have

$$\sum_{i=0}^{v_{m-1}} x_i f_i = \sum_{i=0}^{v_{m-1}} y_{mi} e_i \tag{10}$$

If $m \geqslant k$ and $0 \leqslant i \leqslant v_{m-1}$, then it is easy to show that for any $n \geqslant m$, $v_{n-1} < (n-m)a_n + i < v_n$, p.d. and $n \geqslant k+1$. Thus by Eq. (5), Eq. (7) and Eq. (9) we can obtain $y_{mi} = 0$ whenever $m \geqslant k+2$, $0 \leqslant i \leqslant v_{m-1}$. Hence, using Eq. (10) for $m \geqslant k+2$, we get $\sum_{i=1}^{v_{m-1}} x_i f_i = 0$. Letting $m \to \infty$, we obtain $x = \sum_{i=0}^{\infty} x_i f_i = 0$, which contradicts $x \neq 0$.

Definition 4　For any $x \in E$, if $x = x_0 e_0 + x_1 e_1 + \cdots + x_n e_n$, $n \in Z^+$, then we define $\| x \|_a = | x_0 | + | x_1 | + \cdots + | x_n |$. It is easy to show that $\| \circ \|_a$ is a norm on E.

Lemma 5　$Q_m (m > 2)$ is a bounded linear operator from l_1 into $(E_{(m-1)a_m}, \| \circ \|_a)$, and there is a constant D_m depending only on the numbers $a_1, b_1, \cdots, a_{m-1}, b_{m-1}, a_m$ such that $\| Q_m x \|_a \leqslant D_m \| x \|$ for all $x \in l_1$. p.d.

Proof　Since $E_{(m-1)a_m}$ is a finite dimensional space, the norms $\| \circ \|$ and $\| \circ \|_a$ on $E_{(m-1)a_m}$ are equivalent. Thus by Lemma 2 and Note 2 we obtain Lemma 5.

3　Showing That *T* Has No Non-trivial Closed Invariant Subspaces

Let $d^0(g)$ denote the degree of the polynomial g. For any nonzero polynominal $g(t) = a_0 + a_1 t + \cdots + a_n t^n$, $n \in Z^+$, we write $\mathrm{val}(g) = \min\{i;\ \alpha_i \neq 0\}$. For any vector $y = y_0 e_0 + y_1 e_1 + \cdots + y_n e_n + \cdots$ in l_1, we write $P_n(y) = y_0 e_0 + y_1 e_1 + \cdots + y_n e_n$, $n \in Z^+$.

Theorem 2　Provided d increases sufficiently rapidly, the bounded linear operator T on l_1 has no non-trivial closed invariant subspace.

Proof　It is sufficient to show that for any $x \in l_1 (x \neq 0)$ and any $\varepsilon > 0$, there exists a polynomial q such that $\| q(T)x - e_0 \| < \varepsilon$.

First we construct the polynomial q. Since $a_n \to +\infty$, there is a k in N such that $k \geqslant 3$ and

$$5/a_{k-1} < \varepsilon \tag{11}$$

By Lemma 4, there exist m, r in N such that $m \geqslant r+k$, $Q_m x \neq 0$ and $\mathrm{val}(Q_m x) \leqslant r a_m$. Write $\mathrm{val}(Q_m x) = j$, then $j \leqslant r a_m$. By Note 2, we have $Q_m x \in E_{(m-1)a_m}$. Consequently, we can write

$$Q_m x = y_j e_j + y_{j+1} e_{j+1} + \cdots + y_{(m-1)a_m} e_{(m-1)a_m} \tag{12}$$

where $y_j \neq 0$. Set $h_1(T) = (1/y_j) T^{(r+1)a_m - j}$. Then $h_1(T) Q_m x \in E$ and $(r+1)a_m - j \geqslant a_m$. Noting that $m \geqslant r+k$, $k \geqslant 3$, we have $(r+1)a_m - j \leqslant (m-1)a_m$. Consider two cases separately:

Case 1　If $h_1(T) Q_m x$ is in the form of

$$h_1(T) Q_m x = e_{(r+1)a_m} + y_1^{(1)} e_{(m-1)a_m+1} + y_2^{(1)} e_{(m-1)a_m+2} + \cdots$$

then we set $h(T) = h_1(T)$. Consequently $h(T) Q_m x$ is in the form of $h(T) Q_m x = e_{(r+1)a_m} + z$, where $z \in E$, and $\mathrm{val}(z) > (m-1)a_m$, $\mathrm{val}(h) = (r+1)a_m - j \geqslant a_m$, $d^0(h) \leqslant (m-1)a_m$.

Case 2　If $h_1(T) Q_m x$ is in the form of

$$h_1(T) Q_m x = e_{(r+1)a_m} + y_{i_1}^{(1)} e_{(r+1)a_m + i_1} + \cdots + y_{(m-1)a_m}^{(1)} e_{(m-1)a_m} + \cdots$$

where $y_{i_1}^{(1)} \neq 0$, $i_1 \in N$, and $(r+1)a_m + i_1 \leqslant (m-1)a_m$. Then we set $h_2(T) = h_1(T) - (y_{i_1}^{(1)}/y_j)$

$T^{(r+1)a_m+i_1-j}$. Consequently $\mathrm{val}(h_2)\geqslant a_m$, $d^{(0)}(h_2)\leqslant(m-1)a_m$.

If $h_2(T)Q_mx$ is a vector of the form $h_2(T)Q_mx=e_{(r+1)a_m}+y_1^{(2)}e_{(m-1)a_m+1}+y_2^{(2)}e_{(m-1)a_m+2}+\cdots$, then we take $h(T)=h_2(T)$. Therefore $h(T)Q_mx$ is a vector of the form $h(T)Q_mx=e_{(r+1)a_m}+z$, where $z\in E$, $\mathrm{val}(z)>(m-1)a_m$, $\mathrm{val}(h)\geqslant a_m$, and $d^0(h)\leqslant(m-1)a_m$.

If $h_2(T)Q_mx$ is a vector of the form $h_2(T)Q_mx=e_{(r+1)a_m}+y_{i_2}^{(2)}e_{(r+1)a_m+i_2}+\cdots+y_{(m-1)a_m}^{(2)}e_{(m-1)a_m}+\cdots$, where $y_{i_2}^{(2)}\neq 0$, $i_2>i_1$, $(r+1)a_m+i_2\leqslant(m-1)a_m$, then, continuing in this way (not more than $(m-1)a_m-j$ time), we can obtain a polynomial h such that $h(T)Q_mx=e_{(r+1)a_m}+z$, where $z\in E$, $\mathrm{val}(z)>(m-1)a_m$, $\mathrm{val}(h)\geqslant a_m$ and $d^0(h)\leqslant(m-1)a_m$.

From the above, it can readily be seen that there is a polynomial h such that

$$P_{(m-1)a_m}(h(T)Q_mx)=e_{(r+1)a_m} \tag{13}$$

and $h(T)$ is in the form of

$$h(T)=\lambda_{a_m}T^{a_m}+\lambda_{a_m+1}T^{a_m+1}+\cdots+\lambda_{(m-1)a_m}T^{(m-1)a_m} \tag{14}$$

Set $q(T)=(1/b_m)T^{b_m}h(T)$. Then

$$q(T)=(1/b_m)(\lambda_{a_m}T^{a_m+b_m}+\lambda_{a_m+1}T^{a_m+1+b_m}+\cdots+\lambda_{(m-1)a_m+b_m}T^{(m-1)a_m+b_m}) \tag{15}$$

We now prove $\|q(T)x-e_0\|<\varepsilon$. It follows from the construction of $h(T)$ that the coefficient λ_i of $h(T)$ in Eq. (14) depends only on the coordinate y_i of Q_mx in Eq. (12). Moreover, the coordinate y_i of Q_mx depends only on a_m. Therefore the coefficient λ_i of $h(T)$ depends only on a_m. Write

$$C'_m=|\lambda_{a_m}|+|\lambda_{a_m+1}|+\cdots+|\lambda_{(m-1)a_m}| \tag{16}$$

Then C'_m is a constant depending only on a_m. Note that $\|TQ_mx\|_a=\|Q_mx\|_a$. We obtain by Lemma 5 that

$$\|h(T)Q_mx\|_a\leqslant\sum_{i=a_m}^{(m-1)a_m}|\lambda_i|\,\|T^iQ_mx\|_a=\sum_{i=a_m}^{(m-1)a_m}|\lambda_i|\,\|Q_mx\|_a\leqslant C'_mD_m\|x\| \tag{17}$$

By Lemma 3 and Eq. (15), Eq. (16), we can obtain

$$\|q(T)x-q(T)Q_mx\|\leqslant 2\|x\|C'_m(1/b_m)<1/a_m.\ \mathrm{p.d.} \tag{18}$$

By Eq. (12) and Eq. (15), we have $q(T)Q_mx\in E_{2(m-1)a_m+b_m}$. Therefore we may write $q(T)Q_mx=y_0e_0+y_1e_1+\cdots+y_{2(m-1)a_m+b_m}e_{2(m-1)a_m+b_m}$. Noticing that $2(m-1)a_m+b_m\leqslant 2(a_m+b_m)$, p. d. and $[(1/2)b_m-2(m-1)a_m]/\sqrt{b_m}\geqslant 1$, p. d. we can obtain by Eq. (14), Eq. (15) and Eq. (17) that

$$\|q(T)Q_mx-P_{(m-1)a_m+b_m}(q(T)Q_mx\|\leqslant 1/a_m.\ \mathrm{p.d} \tag{19}$$

If $a_m\leqslant j\leqslant(m-1)a_m$, then it follows from 4) that

$$\|(1/b_m)e_{j+b_m}-e_j\|=1/b_m \tag{20}$$

On the other hand, by Eq. (12) and Eq. (14), We have $\mathrm{val}(h(T)Q_mx)\geqslant a_m$. Hence we may write $h(T)Q_mx=y_{a_m}e_{a_m}+y_{a_m+1}e_{a_m+1}+\cdots$. Therefore it follows from Eq. (13), Eq. (15), Eq. (17) and Eq. (20) that

$$\|P_{(m-1)a_m+b_m}(q(T)Q_mx)-e_{(r+1)a_m}\|<1/a_m.\ \mathrm{p.d.} \tag{21}$$

Since $m\geqslant r+k$, $k\geqslant 3$, it follows that $r+1\leqslant m-1$. Using 8) with n replaced by m, r by $r+1$, and i by $(r+1)a_m$, we obtain

$$\| e_{(r+1)a_m} - e_0 \| \leqslant 2/a_{m-r-1} \tag{22}$$

By Eq.(18)~Eq.(22) and Eq.(11), we can obtain $\| q(T)x - e_0 \| < \varepsilon$. Thus the proof of Theroem 2 is completed.

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关于不变子空间问题的否定解*

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【摘要】 用代数方法进一步简化了不变子空间问题的否定解，在 l_1 上给出了一个不具有非平凡的不变子空间的有界线性算子。特别是用向量的坐标计算代替了某些向量的范数估计，用一些简单的代数运算代替了紧空间技巧。

关　键　词　Banach 空间；　有界线性算子；　不变子空间；　代数；　范数

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