两类含两个变态贝塞尔函数积的积分公式*

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【摘要】 从变态贝塞尔方程出发，构造了它的一个变换形式，利用微积分学中的定理，巧妙地求得了含两个变态贝塞尔函数和的积的一个普遍的积分公式。进而利用变态贝塞尔函数的性质，可简洁地推导出电磁场领域中两类非常重要的积分的解析表达式，在处理圆柱结构下有关电磁场能量或电磁波传输功率的计算问题时，具有普遍的应用价值.

关 键 词 特殊函数；变态贝塞尔函数；积分公式
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柱坐标系下,计算电磁波传输功率或电磁场能量时,常常会遇到含两个贝塞尔函数积的积分问题,其中存在这样两类积分:第一类积分 $\int^{x}[AI_{u}(kx)+BI_{-u}(kx)]\,[A'I_{u}(kx)+B'I_{-u}(kx)]\,\mathrm{d}x/x$ 与第二类积分 $\int^{x}x[AI_{u}(kx)+BI_{-u}(kx)][A'I_{u}(kx)+B'I_{-u}(kx)]\mathrm{d}x$ 。这是两类非常重要的积分,然而对于此类型积分的研究,在已有的关于含贝塞尔函数积分的文献中都没有涉及到[1~4]。本文则从变态贝塞尔方程入手，利用微积分学中的定理,巧妙地求得了含两个变态贝塞尔函数和的积的一个普遍的积分公式,进而利用变态贝塞尔函数的性质，推导出了以上两类积分的解析式。明显地，此积分公式中的系数(A,A',B,B')取不同的值时，即可得到不同的积分表达式，因此本文给出的结果更具有一般性。

1　变态贝塞尔方程的一个变换形式

变态贝塞尔方程

$$\frac{\mathrm{d}^2 f}{\mathrm{d}w^2}+\frac{1}{w}\frac{\mathrm{d}f}{\mathrm{d}w}-(1+\frac{v^2}{w^2})f=0 \tag{1}$$

的解为[1]

$$f=[AI_{v}(w)+BI_{-v}(w)]\text{或}[AI_{v}(w)+BK_{v}(w)]$$

式中　$I_{\pm v}(w)$ 、 $K_{v}(w)$ 分别为第一、二类变态贝塞尔函数。可以证明方程

$$\frac{\mathrm{d}^2 f}{\mathrm{d}w^2}-\frac{2v-1}{w}\frac{\mathrm{d}f}{\mathrm{d}w}-f=0 \tag{2}$$

其解为

$$f=w^{v}[AI_{v}(w)+BI_{-v}(w)]\text{或}f=w^{v}[AI_{v}(w)+BK_{v}(w)]$$

其中　A、B 为任意常数。

取 $w=\varphi(x)$ ，并令 $y=Z(x)f$

于是方程（2）变为

$$\frac{\mathrm{d}^2\{y/Z(x)\}}{\mathrm{d}\{\varphi(x)\}^2}-\frac{2v-1}{\{\varphi(x)\}}\frac{\mathrm{d}\{y/Z(x)\}}{\mathrm{d}\{\varphi(x)\}}-\frac{y}{Z(x)}=0 \tag{3}$$

则 $x=Z(x)\{\varphi(x)\}^{v}[AI_{-v}\{\varphi(x)\}+BI_{-v}\{\varphi(x)\}]$ 就是其一个解。

化简方程（3）得

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\left[-\frac{\varphi''}{\varphi'}-\frac{\varphi'}{\varphi}(2v-1)-2\frac{X'}{X}\right]+y\left[\left\{\frac{\varphi''}{\varphi'}+\frac{\varphi'}{\varphi}(2v-1)+2\frac{X'}{X}\right\}\frac{X'}{X}-\frac{X''}{X}-\{\varphi(x)\}^2\right]=0 \tag{4}$$

定义函数

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$$\frac{\psi'(x)}{\psi(x)}=\frac{\varphi''(x)}{\varphi'(x)}+(2\nu-1)\frac{\varphi'(x)}{\varphi(x)}+2\frac{X'(x)}{X(x)} \tag{5}$$

易知$\psi(x)=\varphi'(x)\{X(x)\}^2\{\varphi(x)\}^{2\nu-1}$满足式(5)，进一步将式(4)中$X'/X, X''/X, X'''/X$代换后可得

$$\frac{d^2y}{dx^2}-\frac{\psi}{\psi}\frac{dy}{dx}+\left[-\frac{1}{2}\frac{\psi\ (x)}{\psi(x)}+\frac{3}{4}\left\{\frac{\psi\ (x)}{\psi(x)}\right\}^2+\frac{1}{2}\frac{\varphi\ \ (x)}{\varphi\ (x)}-\frac{3}{4}\left\{\frac{\varphi\ \ (x)}{\varphi\ (x)}\right\}^2-\left\{\varphi^2(x)+\nu^2-\frac{1}{4}\right\}\left\{\frac{\varphi\ (x)}{\varphi(x)}\right\}^2\right]y=0 \tag{6}$$

若取$\psi(x)\equiv 1$，可以得出方程

$$\frac{d^2y}{dx^2}+\left[\frac{1}{2}\frac{\varphi\ \ (x)}{\varphi\ (x)}-\frac{3}{4}\left\{\frac{\varphi\ \ (x)}{\varphi\ (x)}\right\}^2-\left\{\varphi^2(x)+\nu^2-\frac{1}{4}\right\}\cdot\left\{\frac{\varphi\ (x)}{\varphi(x)}\right\}^2\right]y=0 \tag{7}$$

的通解为$y=\sqrt{\varphi(x)/\varphi'(x)}[AI_\nu\{\varphi(x)\}+BI_{-\nu}\{\varphi(x)\}]$或$\sqrt{\varphi(x)/\varphi'(x)}[AI_\nu\{\varphi(x)\}+BK_\nu\{\varphi(x)\}]$。方程（7）就是我们要寻找的变态贝塞尔方程的一个变形。

2　两类积分解析式的求解

有以下定理（可以通过对式（9）两边取微分而得），若函数y,η满足

$$\frac{d^2y}{dz^2}+py=0\qquad \frac{d^2\eta}{dz^2}+q\eta=0 \tag{8}$$

则

$$\int^z(p-q)y\eta dz=y\frac{d\eta}{dz}-\eta\frac{dy}{dz}+C \tag{9}$$

式中　C是一任意常数。将此结果应用到两个式（7）型的方程，则有积分

$$\begin{aligned}&\int^x\sqrt{\varphi(x)\phi(x)/\{\varphi'(x)\phi'(x)\}}(p-q)[AI_u\{\phi(x)\}+BI_{-u}\{\phi(x)\}][A'I_\nu\{\varphi(x)\}+B'I_{-\nu}\{\varphi(x)\}]dx=\\&\sqrt{\frac{\varphi(x)\phi(x)}{\varphi'(x)\phi'(x)}}\Big\{[AI_u\{\phi(x)\}+BI_{-u}\{\phi(x)\}]\frac{d}{dx}[A'I_\nu\{\varphi(x)\}+B'I_{-\nu}\{\varphi(x)\}]-\\&[A'I_\nu\{\varphi(x)\}+B'I_{-\nu}\{\varphi(x)\}]\frac{d}{dx}[AI_u\{\phi(x)\}+BI_{-u}\{\phi(x)\}]-\\&\frac{1}{2}\left[\frac{\varphi'(x)}{\varphi(x)}-\frac{\varphi''(x)}{\varphi'(x)}-\frac{\phi'(x)}{\phi(x)}+\frac{\phi''(x)}{\phi(x)}\right][AI_u\{\phi(x)\}+BI_{-u}\{\phi(x)\}][A'I_\nu\{\varphi(x)\}+B'I_{-\nu}\{\varphi(x)\}]\end{aligned} \tag{10}$$

式中　I_u, I_ν分别代表u阶ν阶第一类变态贝塞尔函数，$\phi(x),\varphi(x)$是x的任意函数，

$$p=\frac{\phi'''(x)}{2\phi'(x)}-\frac{3}{4}\left\{\frac{\phi''(x)}{\phi'(x)}\right\}^2-\{\phi^2(x)+u^2-1/4\}\left\{\frac{\phi'(x)}{\phi(x)}\right\}^2$$

$$q=\frac{\varphi'''(x)}{2\varphi'(x)}-\frac{3}{4}\left\{\frac{\varphi''(x)}{\varphi'(x)}\right\}^2-\{\varphi^2(x)+u^2-1/4\}\left\{\frac{\varphi'(x)}{\varphi(x)}\right\}^2$$

式(10)是一个普遍的积分表达式，它能为解决许多类含两个变态贝塞尔函数积的积分提供帮助，本文取$\phi(x)=kx$，$\varphi(x)=lx$，则

$$p-q=(-k^2+l^2)-(u^2-\nu^2)/x^2$$

积分式（10）化简为

$$\begin{aligned}&\int^x-\{(k^2-l^2)x+(u^2-\nu^2)/x\}[AI_u(kx)+BI_{-u}(kx)][A'I_\nu(lx)+B'I_{-\nu}(lx)]dx\\&=x\{l[AI_u(kx)+BI_{-u}(kx)][A'I_{\nu+1}(lx)+B'I_{-\nu-1}(lx)]-k[A'I_\nu(lx)+B'I_{-\nu}(lx)][AI_{u+1}(kx)+BI_{-u-1}(kx)]\}\\&\quad-(u-\nu)[AI_u(kx)+BI_{-u}(kx)][A'I_\nu(lx)+B'I_{-\nu}(lx)]\end{aligned} \tag{11}$$

利用式(11)，就可以导出在前文中提出的两类积分的解析表达式。

2.1　$\int^x[AI_\nu(kx)+BI_{-\nu}(kx)][A\ I_\nu(kx)+B\ I_{-\nu}(kx)]\frac{dx}{x}$型积分的解析式

在式(11)中，令$l=k$，显然有

$$\int^{x}[AI_{u}(kx)+BI_{-u}(kx)][A'I_{v}(lx)+B'I_{-v}(lx)]\frac{dx}{x}=$$

$$\{kx\{[AI_{u}(kx)+BI_{-u}(kx)][A'I_{v+1}(kx)+B'I_{-v-1}(kx)]-[A I_{v}(kx)+BI_{-v}(kx)][AI_{u+1}(kx)+BI_{-u-1}(kx)+C\}/(v^{2}-u^{2})+$$

$$[AI_{u}(kx)+BI_{-u}(kx)][A'I_{v}(lx)+B'I_{-v}(lx)/(v+u) \qquad (12)$$

在此，取 $C=-[AB\ \ +A\ B]\cdot 2\sin(u\pi)/\pi$ ，于是当 $v\to u$ 时，积分右边第一式的分子等于零，而分母也为零，则应用罗比达法则得

$$\int^{x}[AI_{u}(kx)+BI_{-u}(kx)][A\ I_{u}(kx)+B\ I_{-u}(kx)]\frac{dx}{x}=\frac{kx}{2u}\left\{[AI_{u}(kx)+BI_{-u}(kx)]\frac{\partial}{\partial u}[A\ I_{u+1}(kx)+B\ I_{-(u+1)}(kx)]\right.$$

$$\left.-[AI_{u+1}(kx)+BI_{-(u+1)}(kx)]\frac{\partial}{\partial u}[A\ I_{u}(kx)+B\ I_{-u}(kx)]\right\}+\frac{1}{2u}[AJ_{u}(kx)+BJ_{-u}(kx)][A'J_{u}(kx)+B'J_{-u}(kx)] \qquad (13)$$

特例　当 $A=B'=1,B=A'=0$ 时，

$$\int^{x}I_{u}(kx)I_{-u}(kx)]\frac{dx}{x}=\frac{kx}{2u}\left\{I_{u}(kx)\frac{\partial}{\partial u}[I_{-(u+1)}(kx)]-I_{u+1}(kx)\frac{\partial}{\partial u}[I_{-u}(kx)]\right\}-\frac{1}{2u}[I_{u}(kx)I_{-u}(kx)] \qquad (14)$$

当 $A=A'=1,B=B'=0$ 时

$$\int^{x}I_{u}(kx)I_{u}(kx)]\frac{dx}{x}=\frac{kx}{2u}\left\{I_{u}(kx)\frac{\partial}{\partial u}[I_{u+1}(kx)]-I_{u+1}(kx)\frac{\partial}{\partial u}[I_{u}(kx)]\right\}+\frac{1}{2u}[I_{u}(kx)I_{u}(kx)] \qquad (15)$$

2.2 $\int^{x}x[AI_{u}(kx)+BI_{-u}(kx)][A\ I_{u}(kx)+B\ I_{-u}(kx)]dx$ 型积分的解析式

在式(11)中,令 $v=u$,则化为

$$\int^{x}x\cdot[AI_{u}(kx)+BI_{-u}(kx)][A'I_{v}(lx)+B'I_{-v}(lx)]dx=$$

$$\frac{-x}{k^{2}-l^{2}}\{l[AI_{u}(kx)+BI_{-u}(kx)][A'I_{u+1}(lx)+B'I_{-u-1}(lx)]-kx[A'I_{u}(lx)+B'I_{-u}(lx)][AI_{u+1}(kx)+BI_{-u-1}(kx)]\} \qquad (16)$$

当 $l\to k$ 时，右边分子分母都为零，同样地，利用罗比达法则对 l 求导可得

$$\int^{x}x[AI_{u}(kx)+BI_{-u}(kx)][A'I_{v}(lx)+B'I_{-v}(lx)]dx=\frac{x}{2k}\{[AI_{u}(kx)+BI_{-u}(kx)][A'I_{u+1}(lx)+B'I_{-u-1}(lx)]+$$

$$kx[AI_{u}(kx)+BI_{-u}(kx)][A'I_{u+1}'(kx)+B'I_{-u-1}'(kx)]-kx[A'I_{u}'(kx)+B'I_{-u}'(kx)][AI_{u+1}(kx)+BI_{-u-1}(kx)]\} \qquad (17)$$

式中

$$I_{\pm u}'(kx)=\frac{d}{d(kx)}\{I_{\pm u}(kx)\}$$

应用变态贝塞尔函数的递推公式[5],式(17)右边等于

$$\frac{x}{2k}\{[AI_{u}(kx)+BI_{-u}(kx)][A'I_{u+1}(lx)+B'I_{-u-1}(lx)]+$$

$$kx[AI_{u}(kx)+BI_{-u}(kx)][A'\{I_{u}(kx)-(u+1)/kx\cdot I_{u+1}(kx)\}+B'\{I_{-u}(kx)-(u+1)/kx\cdot I_{-u-1}(kx)\}]-$$

$$kx[A'\{I_{u+1}(kx)+u/kx\cdot I_{u}(kx)\}+B'\{I_{-u-1}(kx)+u/kx\cdot I_{-u}(kx)\}][AI_{u+1}(kx)+BI_{-u-1}(kx)]\} \qquad (18)$$

经过多次代换，最后得到

$$\int^{x}x[AI_{u}(kx)+BI_{-u}(kx)][A'I_{u}(kx)+B'I_{-u}(kx)]dx=\frac{x^{2}}{4}\{2[AI_{u}(kx)+BI_{-u}(kx)][A'I_{u}(kx)+B'I_{-u}(kx)-]$$

$$[AI_{u-1}(kx)+BI_{-(u-1)}(kx)][A'I_{u+1}(kx)+B'I_{-(u+1)}(kx)]-[AI_{u+1}(kx)+BI_{-(u+1)}(kx)][A'I_{u-1}(kx)+B'I_{-(u-1)}(kx)]\} \qquad (19)$$

特别地，当 $A=A\ \ =1,B=B\ \ =0$ 时

$$\int^{x}xI_{u}^{2}(kx)dx=\frac{x^{2}}{2}\{I_{u}^{2}(kx)-I_{u+1}(kx)I_{u-1}(kx)\}=-\frac{x^{2}}{2}\{I_{u}'^{2}(kx)-(1+u^{2}/(kx)^{2})I_{u}^{2}(kx)\} \qquad (20)$$

这与文献[1] 中的结果相同。当 $A=B\ \ =1,B=A\ \ =0$ 时，

$$\int^{x}xI_{u}(kx)I_{-u}(kx)dx=\frac{x^{2}}{4}\{2I_{u}(kx)I_{-u}(kx)-I_{u+1}(kx)I_{-(u-1)}(kx)-I_{-(u+1)}(kx)I_{u-1}(kx)\} \qquad (21)$$

通过以上的分析及变换，我们得到了这两类型积分的解析表达式。显然，当式(12)、(19)中的系数(A,A',B,B')取不同的值时，即可给出各种积分特例。其中的一些特例(如式（19）)就是已有的积分公式，

这也间接地证明了本文所得的积分普适公式的正确性，因此这些积分公式完全可以应用到电磁领域内有关数学问题的解决之中。

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Formulae of Two Kinds of Integrals Involving Product of Two Modified Bessel Functions

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Abstract A new transformation of modified Bessel function is built in this paper. By applying the result of the theorem on the differential and integral calculus to any two equation of the type of this transformation smartly, a general integral formula involving the product of two modified Bessel functions is obtained. Furthermore, by recurrence relations of modified Bessel function, we get the analytical expressions of two kinds of integrals are gotten respectively, which can be widely used in the evaluation of the power in the electromagnetic fields.

Key words special function; modified bessel function; integral formulae