第29卷 第2期 电子科技大学学报 Vol.29 No.2
2000年4月 Journal of UEST of China Apr. 2000

Enumeration of Some Circular Graphs*

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Abstract The structures of circular graphs of degree 4 are discussed. All non-isomorphic connected circular graphs with order n and degree 4 are divided into two types: type I and type II. A formula calculating the number of type I is given. The formula about enumeration of circular graphs of non-isomorphic with order n and degree 4 is improved.

Key words graph; circular graph; enumeration; isomorphism

1 Introducation

The terminology and notation in this paper are similar to Ref.[1]. All graphs discussed here are finite and simple.

Let $Z_n=\{0,1,2,\cdots,n-1\}$, $S\subseteq Z_n-\{0\}$, $-S=S \pmod n$, namely there exist $j_1,j_2,\cdots,j_r$ such that $S=\{j_1,j_2,\cdots,j_r,n-j_1,n-j_2,\cdots,n-j_r\}$. S is called characteristic set.

Definition 1 The graph G with order n is called circular graph if it satisfies:

(i) $V(G)=Z_n$;

(ii) $E(G)=\{(i,j)\mid j-i\in S\}$,where the operation takes module n.

The graph G in definition 1 is denoted by $C_n\langle j_1,j_2,\cdots,j_r\rangle$, where $j_1<j_2<\cdots<j_r$, $j_1,j_2,\cdots,j_r$ are called spanning elements.

The structures of circular graphs having degree 3 have been given. Some upper bounds of number of the non-isomorphic circular graphs having order n and degree k have also been obtained. But, the accurate number is not given. In this paper, we discuss the structures of circular graphs of degree 4. Then we improve the formula in Ref.[2] about enumeration of circular graphs with order n and degree 4.

Let gcd(x,y) be the maximum common divisor of x,y. It is proved that a circular graph $C_n\langle j_1,j_2,\cdots,j_r\rangle$ is a connected graph if and only if $\gcd(n,j_1,j_2,\cdots,j_r)=1$ or $\gcd(m_1,m_2,\cdots,m_r)=1$ where $m_i=\gcd(n,j_i)$. It is also proved that if G is a non-connected circular graph with degree k, then any two connected components of G are isomorphic circular graphs of degree k. So, we only discuss connected circular graph.

Given a positive integer n ($n\geqslant 4$).Let

$$f(n)=\begin{cases}(n/2)-1 & n \text{ is even}\\ (n-1)/2 & n \text{ is old}\end{cases}$$

and $G_n=\{C_n\langle u,v\rangle \mid u<v;\ u,v\in\{1,2,\ldots,f(n)\}\text{ and }\gcd(n,u,v)=1\}$. G_n contains all connected circular graphs with order n and degree 4. But, for any $G_n\langle i,j\rangle, C_n\langle s,t\rangle\in G_n$, $G_n\langle i,j\rangle$ and $C_n\langle s,t\rangle$ may be isomorphic. The cardinality of a set S is denoted by $|S|$, $\lfloor m\rfloor$ denotes the smallest integer$\geqslant m$. Suppose $n=p_1^{\alpha(1)}p_2^{\alpha(2)}\ldots p_k^{\alpha(k)}$ where $p_1,p_2,\cdots,p_k$ are the distinct prime divisors of n, we have

$$|G_n|=\binom{f(n)}{2}-\sum_{i=1}^{k}\binom{X_i}{2}+\sum_{i=1}^{k}\sum_{j>i}\binom{X_{i,j}}{2}-\sum_{i=1}^{k}\sum_{j>i}\sum_{h>j}\binom{X_{i,j,h}}{2}+\cdots+(-1)^k\binom{X_{1,2,\cdots,k}}{2}$$

where $X_{1,2,\cdots,r}=\left\lfloor\dfrac{f(n)}{p_1p_2\cdots p_r}\right\rfloor$.

Received October 4, 1999

* The project supported by Electronic Industrial Ministry Advanced Research Foundation

For integer x,y,z, if $x\equiv a \pmod n$, $0\leqslant a<n$, then let

$$<x>_n^*=\begin{cases} a & 0\leqslant a<\lfloor n/2\rfloor \\ n-a & a\geqslant\lfloor n/2\rfloor \end{cases}$$

Clearly $\langle x\rangle^*_n\in\{0,1,\cdots,\lfloor n/2\rfloor\}$ and $x=tn\pm a$. Let $N_n=\{0,1,\cdots,\lfloor n/2\rfloor\}$, $M_n=\{q\mid \gcd(n,q)=1, 1\leqslant q<n/2\}$ and $m=|M_n|$. Clearly $m=\varphi(n)/2$, where $\varphi(n)$ is Euler function, *i.e*, $\varphi(n)$ is the number of integers between 1 and n which are relatively prime to n. We define operator $*$ in N_n as follows

$$x*y=\langle xy\rangle^*_n \qquad \text{for all } x,y\in N_n$$

Lemma 1 [2] For all $x,y,z\in N_n$ and $q\in M_m$ we have

(i) $(x*y)*z=x*(y*z)$;

(ii) if $q*x=q*y$, then $x=y$.

Given $C_n\langle i,j\rangle\in G_n$, let

$$A(i,j)=\{C_n\langle u,v\rangle \mid C_n\langle u,v\rangle\in G_n \text{ and } C_n\langle u,v\rangle\cong C_n\langle i,j\rangle\}$$

where $C_n\langle u,v\rangle\cong C_n\langle i,j\rangle$ means that the graphs $C_n\langle u,v\rangle$ and $C_n\langle i,j\rangle$ are isomorphic. Clearly, the relation $\cong$ is a equivalent relation in G_m and $A(i,j)$ is a equivalence class of the equivalent relation.

Let $\langle x,y\rangle^*_n=\langle b,c\rangle$, where $\{b,c\}=\{\langle x\rangle^*_n,\langle y\rangle^*_n\}$ and $b<c$.

Definition 2 [2] Let $A(u,v)$ be a equivalence class in G_n for the relation $\cong$ and $M_n=\{q_1,q_2,\cdots,q_m\}$. $A(u,v)$ is called a II-equivalence class, if $\langle q_1u,q_1v\rangle^*_n$, $\langle q_2u,q_2v\rangle^*_n$, $\cdots$, $\langle q_mu,q_mv\rangle^*_n$ are all distinct. Otherwise, $A(u,v)$ is called a I-equivalence class.

Ref. [2] shows that a equivalence class $A(u,v)$ is a I-equivalence class if and only if there exists $q\in M_n-\{1\}$ such that $\{q*u,q*v\}=\{u,v\}$. Let

$$G^*_n=\{A(u,v)\mid C_n\langle u,v\rangle\in G_n\}$$

Clearly, $|G^*_n|$ is the number of circular graphs with order n and degree 4. Ref.[2] shows

$$|G_n^*|=\frac{|G_n|-T\dfrac{m}{2}}{m}+T \tag{1}$$

where $T=|\{A(u,v)\mid A(u,v)\in G^*_n \text{ and } A(u,v) \text{ is a I-equivalence class}\}|$ and $T=0$ if m is odd. But, how to compute T is not given by Ref.[2]. In the following, we determine the structure of a circular graph which is a I-equivalence class, then give out a formula about T.

2 Main Results

Definition 3 Let $C_n\langle k_1,k_2\rangle$ be a generator of equivalence class A. $C_n\langle k_1,k_2\rangle$ is called a min-generator if $k_1=\min\{x_1\mid C_n\langle x_1,x_2\rangle\in A\}$, $k_2=\min\{x_2\mid C_n\langle k_1,x_2\rangle\in A\}$.

Lemma 2 [3] $C_n\langle k_1,k_2\rangle\cong C_n\langle k_1{}'k_2{}'\rangle$ if and only if there exists a $q\in M_n$ such that $\langle qk_1,qk_2\rangle^*_n=\langle k_1{}',k_2{}'\rangle$.

Lemma 2 implies that $A(u,v)=\{C_n\langle x,y\rangle \mid \langle x,y\rangle=\langle qu,qv\rangle^*_m\ q\in M_n\}$.

Lemma 3 For $k\in N_n-\{0\}$, let $s=\gcd(k,n)$. Then $s\leqslant\langle qk\rangle^*_n$ for all $q\in M_n$.

Proof Since $s=\gcd(k,n)$, there exist two positive integers k' and n' such that $k=k's$ and $n=n's$. Suppose $qk=an\pm r$ where a is a integer and $r\in N_n$. So, $qk-an=\pm r\Rightarrow qk's-an's=\pm r\Rightarrow s(qk'-an')=\pm r\Rightarrow s\leqslant r$, *i.e.* $s\leqslant\langle qk\rangle^*_n$ ⌈

Lemma 4 $C_n\langle u,v\rangle$ is a min-generator of equivalence class A if and only if $u\mid n$ where $1\leqslant u<v<n/2$ and $\gcd(n,u,v)=1$.

Proof Let

$$u=\min_{C_n\langle k',k''\rangle\in A}\{\gcd(k',n),\gcd(k'',n)\} \tag{2}$$

Suppose $u = \gcd(k_1,n)$, correspondingly, $C_n\langle k_1,k_2\rangle \in A$. Since $\gcd(u,n) = u = \gcd(k_1,n)$, we have $C_n\langle u\rangle \cong C_n\langle k_1\rangle$. According to Lemma 2 (for circular graphs of degree 2, Lemma 2 is also true), there exists a $q\in M_n$ such that

$$\langle qk_1\rangle^*_n = u \tag{3}$$

Let

$$\langle qk_2\rangle^*_n = v' \tag{4}$$

we have, for Eq.(2) and Lemma 3, $u \leqslant \gcd(k_2,n) = \gcd(qk_2,n) \leqslant \langle qk_2\rangle^*_n = v$.

By Eqs.(3),(4) and Lemma 2, $C_n\langle k_1,k_2\rangle \cong C_n\langle u,v'\rangle$. Hence $C_n\langle u,v'\rangle \in A$. For all $C_n\langle k',k''\rangle \in A$, by Eq.(2),we have

$$u \leqslant \gcd(k',n) \leqslant k'$$

$$u \leqslant \gcd(k'',n) \leqslant k''$$

Let $v = \min\{v' \mid C_n\langle u,v'\rangle \in A\}$. Then $C_n\langle u,v\rangle$ is a min-generator of A.

For all $C_n\langle k',k''\rangle \in A$, if $k' \mid n$ is not true, then $k_1 > \gcd(k_1,n)$.So, $u \leqslant \gcd(k_1,n) < k'$. It follows that $C_n\langle k',k''\rangle$ is not a min-generator of A □

Lemma 5 Let $1 \leqslant k_1 < k_2 < n/2$, $k_1|n$, $k_2|n$, $\gcd(k_1,k_2) = 1$, then

(i) $C_n\langle k_1,k_2\rangle$ is a min-generator of the equivalence class $A(k_1,k_2)$;

(ii) if $A(k_1,k_2)$ is a I-equivalence class, then $k_1k_2=n/2$ or $k_1k_2=n$.

Proof (i) For any $C_n\langle x,y\rangle \in A(k_1,k_2)$, there exists a $q\in M_n$ such that $\langle qk_1,qk_2\rangle^*_n = \langle x,y\rangle$. By Lemma 3, $k_1 \leqslant \langle qk_1\rangle^*_n$, $k_2 \leqslant \langle qk_2\rangle^*_n$, $k_1<k_2$. Hence $k_1 \leqslant \min\{\langle qk_1\rangle^*_n, \langle qk_2\rangle^*_n\} = \min\{x,y\}$.

Since $\langle qk_2\rangle^*_n \geqslant k_2 > k_1$ for all $q\in M_n$, $C_n\langle k_1,k_2\rangle$ is a min-generator of $A(k_1,k_2)$.

(ii) Since $k_1|n$, $k_2|n$, $\gcd(k_1,k_2) = 1$, we have $n = n'k_1k_2$, where n' is a positive integer. $C_n\langle k_1,k_2\rangle$ is a I-equivalence class. So, there exists a $q\in M_n$ and $q\neq 1$ such that $\langle qk_1,qk_2\rangle^*_n = \langle k_1,k_2\rangle$. It implies that

$$qk_1 = an \pm k_1 \tag{5}$$

$$qk_2 = bn \pm k_2 \tag{6}$$

Case 1 $qk_1 = an + k_1$ and $qk_2 = bn + k_2$.

According to $\gcd(k_1,k_2) = 1$,we have $xk_1 + yk_2 = 1$. So, $q = q(xk_1 + yk_2) = xqk_1 + yqk_2 = x(an+k_1) + y(bn+k_2) = (xa+yb)n + (xk_1+yk_2) = (xa+yb)n + 1 \Rightarrow q\equiv 1 \pmod n \Rightarrow q=1$(because $q\in M_n$). It contradicts $q\neq 1$. Hence this case does not occur.

Case 2 $qk_1 = an-k_1$ and $qk_2=bn-k_2$. Similar to case 1, this case does not occur.

Case 3

$$qk_1 = an + k_1 \tag{7}$$

$$qk_2 = bn-k_2 \tag{8}$$

By Eq.(7)$\times k_2$ and Eq.(8)$\times k_2$, we obtain $2k_1k_2 = ank_2 - bnk_1 \Rightarrow k_1k_2 = (ak_2 - bk_1)(n/2) \Rightarrow k_1k_2 \equiv 0 \pmod{n/2}$.

Case 4 $qk_1 = an-k_1$ and $qk_2 = bn + k_2$. Similar to case 3, we have $k_1k_2 \equiv 0 \pmod{n/2}$.

Since $n = n'k_1k_2$ and $k_1 < k_2$, we have $1 < k_1k_2 \leqslant n$. So, $k_1k_2 = n/2$ or $k_1k_2 = n$ □

Lemma 6 Suppose $1 \leqslant k_1 < l < k_2 < n/2$, $\gcd(n,k_1) = k_1$, $\gcd(n,k_2) = l$ and $\gcd(k_1,k_2) = 1$. If $C_n\langle k_1,k_2\rangle$ belongs to a I-equivalence class, then $C_n\langle k_1,k_2\rangle \cong C_n\langle k_1,l\rangle$.

Proof Since $\gcd(n,k_2) = l$, there exists a integer u such that $k_2 = lu$. Clearly, $\gcd(n,u) = 1$. Consider $1 < u < n/2$. So, $u\in M_n$. It follows that $k_2 = lu = l*u$ (because $1 < k_2 < n/2$).

Since $C_n\langle k_1,k_2\rangle$ belongs to a I-equivalence class, there exists a $q\in M_n$, $q\neq 1$ such that $\langle qk_1, qk_2\rangle^*_n = \langle k_1, k_2\rangle \Rightarrow q*k_2 = k_2 \Rightarrow q*(l*u) = l*u \Rightarrow (q*l)*u = l*u \Rightarrow q*l = l$ (according to Lemma 1) $\Rightarrow \langle qk_1,ql\rangle^*_n = \langle k_1,l\rangle \Rightarrow$

$C_n\langle k_1,l\rangle$ is a min-generator of the I-equivalence class $A(k_1,l) \Rightarrow k_1 l = n/2$ or $k_1 l = n$ (by Lemma 5) $\Rightarrow 2k_1 l \equiv 0 \pmod n$.

Since $\gcd(k_1,l) = 1$, there exist integers x,y such that $xk_1 - yl = 1$. It follows that

$$(xk_1 + yl) - 2yl = 1$$
$$-(xk_1 - yl) + 2xk_1 = 1$$

Let $q_0 = xk_1 + yl$, we obtain

$$q_0 - 2yl = 1 \tag{9}$$
$$-q_0 + 2xk_1 = 1 \tag{10}$$

So, $\gcd(q_0,l) = 1$ and $\gcd(q_0, 2k_1) = 1 \Rightarrow \gcd(n,q_0) = 1$.

By Eqs.(9) and (10), $(q_0-1)k_1 = 2lk_1 y$ and $(q_0+1)l = 2lk_1 x \Rightarrow (q_0-1)k_1 \equiv 0 \pmod n$ and $(q_0+1)l \equiv 0 \pmod n$ because $2lk_1 \equiv 0 \pmod n$. So, $q_0 k_1 \equiv k_1 \pmod n$ and $q_0 l \equiv n-l \pmod n$.

Let $q' = \langle q_0\rangle^*_n$. We have $\gcd(q',n) = \gcd(q_0, n) = 1$, $q' \in M_n$ and $q' \neq 1$. On the other hand, $q'k_1 \equiv k_1 \pmod n$ and $q'l \equiv n-l \pmod n$. Hence $C_n\langle k_1, k_2\rangle \cong C_n\langle k_1, l\rangle$ □

Theorem 1 Let $1 \leqslant k_1 < k_2 < n/2$, $\gcd(n,k_1) = k_1$, and $\gcd(k_1,k_2) = 1$. Then, $C_n\langle k_1,k_2\rangle$ is a min-generator of the I-equivalence class $A(k_1, k_2)$ if and only if

(i) if $k_1 = 1$, then $\langle k_2^2\rangle^*_n = 1$ and $k_2 \in M_n$;

(ii) if $k_1 > 1$, then $k_1 k_2 = n/2$ or $k_1 k_2 = n$.

Proof Suppose $C_n\langle k_1, k_2\rangle$ is a min-generator of $A(k_1,k_2)$. Let $\gcd(n,k_2) = l$:

Case 1 $l = k_2$. By (ii) of Lemma 5, we have $k_1 k_2 = n/2$ or $k_1 k_2 = n$. Obviously, $k_1 > 1$.

Case 2 $l < k_2$ and $k_1 < l$. By Lemma 6, $C_n\langle k_1,k_2\rangle \cong C_n\langle k_1,l\rangle$. Thus, $C_n\langle k_1,l\rangle$ is a min-generator of $A(k_1,k_2)$. It is contradiction because $C_n\langle k_1,k_2\rangle$ is a min-generator of $A(k_1,k_2)$. So, this case does not occur .

Case 3 $l < k_2$ and $k_1 = l$. In this case $\gcd(n,k_1,k_2) = k_1$. Since $\gcd(k_1,k_2) = 1$, we have $k_1 = 1$.

Since $A(k_1,k_2)$ is a I-equivalence class, there exists a $q \in M_n$ $(q \neq 1)$ such that $\langle qk_1,qk_2\rangle^*_n = \langle k_1,k_2\rangle$, *i.e.* $\langle q,qk_2\rangle^*_n = \langle 1,k_2\rangle$. Hence $k_2 = q$ and $\langle k_2^2\rangle^*_n = 1$.

Conversely, suppose (i) and (ii) are true. For (ii), by (i) in Lemma 5, $C_n\langle k_1, k_2\rangle$ is a min-generator of $A(k_1,k_2)$.

For (i), since $k_2 \in M_n$ and $k_2 > 1$, we have $\langle k_2 k_1, k_2 k_2\rangle^*_n = \langle k_2, k_2^2\rangle^*_n = \langle 1,k_2\rangle = \langle k_1, k_2\rangle$. So, $C_n\langle k_1, k_2\rangle$ is a min-generator of $A(k_1,k_2)$ □

The structure of a I-equivalence class is given by Theorem 1. The number of the I-equivalence classes will be obtained by the following theorem.

Theorem 2 Let T denote the number of I-equivalence in G_n, and $n = p_1^{\alpha(1)} p_2^{\alpha(2)} \cdots p_k^{\alpha(k)}$ where $p_1, p_2, \cdots, p_k$ are the distinct prime divisors of n. Then, $T = T_1 + T_2$, where T_1 denotes the number such that $\langle q^2\rangle^*_n = 1$ in $M_n - \{1\}$,and

$$T_2 = \begin{cases} 3(2^{k-2}) - 1 & n \text{ is even and } n/2 \text{ is odd} \\ 2^k - 2 & n \text{ is even and } n/2 \text{ is even} \\ 2^{k-1} - 1 & n \text{ is odd} \end{cases} \tag{11}$$

Proof By Lemma 4, the min-generator $C_n\langle k_1,k_2\rangle$ of I-equivalence class should satisfy $k_1 | n$. Since the graph $C_n\langle k_1,k_2\rangle$ is connected, we have $\gcd(k_1,k_2) = 1$. So, T should be computed in terms of (i) and (ii) in Theorem 1. It is easy to say that the number of I-equivalence classes which satisfy (i) in Theorem 1 is T_1. Let T_2 be the number of I-equivalence classes which satisfy (ii) in Theorem 1. First, the number of distinct pair $\{k_1,k_2\}$ such that $k_1 k_2 = n$ and (ii) in Theorem 1 is $2^{k-1}-1$. This is because the number partitioning the k-set $\{p_1^{\alpha(1)}, p_2^{\alpha(2)}, \cdots, p_k^{\alpha(k)}\}$ into two blocks is $2^{k-1}-1$. Similarly, for $n/2$ even the number of distinct $\{k_1,k_2\}$

such that $k_1k_2 = n/2$ is also $2^{k-1}-1$. For n even and $n/2$ odd, since we have $n/2=p_2^{\alpha(2)}p_3^{\alpha(3)}\dots p_k^{\alpha(k)}$, the number of distinct $\{k_1,k_2\}$ such that $k_1k_2 = n/2$ is $2^{k-2}-1$. So, we can obtain Eq.(11) by simple computation □

As mentioned above, we obtain a computational formula and method about T of Eq.(1). Therefore the formula (1) is improved.

Example　For $n = 2^3\times 3 = 24$, we have $|G_{24}| = 42$, $M_{24}= \{1,5,7,11\}$ and $m = 4$. Thus $T_2= 2^k-2 = 2^2-2 = 2$. Since $\langle 5^2\rangle^*_{24}= \langle 7^2\rangle^*_{24}= \langle 11^2\rangle^*_{24}=1$, we obtain $T_1=3$. Thus $T=2+3=5$. Therefore $|G^*_{24}| = 8+5 =13$.

Since the circular graphs of degree 5 can be obtained by adding the generator $n/2$ (n is even) to a circular graph of degree 4, the structure and the number of the circular graphs of degree 5 can be discussed analogously.

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一类循环图的计数*

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【摘要】 讨论了 4 度循环图的结构。所有 n 阶 4 度非同构的连通循环图被分为两类：I 类和 II 类。给出了计算 I 类循环图的计算公式，改进了关于 n 阶 4 度非同构的循环图的计数公式。

关 键 词　图；循环图；计数；同构

中图分类号　O157.5

1999 年 10 月 4 日收稿

* 电子部预研基金资助项目

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