多维寿命分布模型

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【摘要】针对两部件组成系统的二维寿命分布，提出了基于条件失效率函数刻画方法，这是一种刻画寿命分布有效的新途径。同时，推广了条件失效率函数，得到了n维寿命分布的刻画情况。

关 键 词 多维寿命分布; 条件失效率函数; 联合概率密度函数; 相依性
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Multivariate Life Distribution Model

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Abstract In this paper, an effective new way has been proposed to depict the bivariate life distribution based on conditional failure rate functions. And these conditional failure rate functions have been extended to generalize the depiction in n-dimensional life distribution.

Key words multivariate life distribution; conditional failure rate function; joint density function; dependence

可靠性数学理论中，分析由若干部件组成的系统的可靠性问题时，涉及到部件寿命分布之间种种复杂的相依关系时，由于对相依性缺乏有效的定量刻画手段，往往也只能过分牵强地假定其组成部件的寿命分布是相互独立的[1]，但这种假定在许多情况下并不符合实际。这种假定被广泛采用，一方面在于它使系统可靠性的分析相对简单、在数学上易于处理；另一方面因为传统可靠性理论的基本数学工具—概率论，对随机变量相依性的机制缺乏有效的刻画手段。

然而由于文献[2]提出的两类条件失效函数有很好的直观意义且易获得。鉴于此，本文通过所拓展的条件失效率函数刻画了多维寿命分布，给出了刻画可靠性理论中相依性的一种有效建模新途径。

1 两类条件失效率函数

(1) 第一类条件失效率函数为：

$$r_i(t_i \mid X_j > t_j) \underset{\Delta t_i \to 0}{\overset{\Delta}{=}} \lim \frac{P\{t_i < X_i \le t_i + \Delta t_i \mid X_i > t_i, X_j > t_j\}}{\Delta t_i} = \frac{\int_{t_j}^{\infty} f(t_i, x_j)\mathrm{d}x_j}{P\{X_i > t_i, X_j > t_j\}} = \frac{f_i(t_i \mid X_j > t_j)}{P(X_i > t_i \mid X_j > t_j)} \tag{1}$$

式(1)表示的意义是：部件i，j分别工作到时刻t_i，t_j均仍然正常的条件下，部件i在(t_i，$t_i+\Delta t_i$] 中失效的概率与Δt_i之比的极限。

(2) 第二类条件失效率函数为：

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$$h_i(t_i \mid X_j = t_j) \underset{\Delta t_i \to 0}{\triangleq \lim} \frac{P\{t_i < X_i \le t_i + \Delta t_i \mid X_i > t_i, X_j = t_j\}}{\Delta t_i} = \frac{f(t_i, t_j)}{\int_{t_i}^{\infty} f(x_i, t_j)\mathrm{d}x_i} = \frac{f_i(t_i \mid X_j = t_j)}{P(X_i > t_i \mid X_j = t_j)} \tag{2}$$

式(2)表示的是：部件j工作到时刻t_j已失效及部件i工作到时刻t_i仍然正常的条件下，部件i在$(t_i$，$t_i+\Delta t_i]$ 中失效的概率与Δt_i之比的极限。以上均有 $i, j = 1, 2$。

注：以下讨论的分布函数都是假定绝对连续的。

2　主要结论

2.1　二维的情况

定理 1　如果f是非负随机变量(X_1, X_2)的联合概率密度函数，那么：

$$f(t_1, t_2) = r_{i_1}(t_{i_1} \mid X_{i_2} > t_{i_1}) h_{i_2}(t_{i_2} \mid X_{i_1} = t_{i_1}) \exp\{-\int_0^{t_{i_1}} \sum_{j=1}^{2} r_{i_j}(u \mid X_{3-i_j} > t_{i_1})\mathrm{d}u - \int_{t_{i_1}}^{t_{i_2}} h_{i_2}(u \mid X_{i_1} = t_{i_1})\mathrm{d}u\} \tag{3}$$

式中　随机变量 X_{i_1}, X_{i_2} 的观察值分别为 t_{i_1}, t_{i_2}；(i_1, i_2) 是(1,2)满足 $0 \le t_{i_1} \le t_{i_2}$ 的某个排列。

证明：不失一般性，设 $i_1 = 1,\ \ i_2 = 2$，即为$0 \le t_1 \le t_2$：

$$f(t_1, t_2) = \lim_{\Delta t_2 \to 0} \lim_{\Delta t_1 \to 0} \frac{P\{t_1 < X_1 \le t_1 + \Delta t_1, t_2 < X_2 \le t_2 + \Delta t_2\}}{\Delta t_1 \Delta t_2} \tag{4}$$

$$\begin{aligned} P\{t_1 < X_1 \le t_1 + \Delta t_1, t_2 < X_2 \le t_2 + \Delta t_2\} = P\{X_2 > t_1, X_1 > t_1\} P\{t_1 < X_1 \le t_1 + \Delta t_1 \mid X_2 > t_1, X_1 > t_1\} \times \\ P\{X_2 > t_2 \mid t_1 < X_1 \le t_1 + \Delta t_1, X_2 > t_1\} P\{t_2 < X_2 \le t_2 + \Delta t_2 \mid t_1 < X_1 \le t_1 + \Delta t_1, X_2 > t_2\} \end{aligned} \tag{5}$$

式(5)两边除以 $\Delta t_1 \Delta t_2$，令 $\Delta t_1 \to 0, \Delta t_2 \to 0$ 并由式(1)、(2)及(4)、(5)可得：

$$f(t_1, t_2) = R(t_1, t_1)\, r_1(t_1 \mid X_2 > t_1) P\{X_2 > t_2 \mid X_1 = t_1, X_2 > t_1\} h_2(t_2 \mid X_1 = t_1) \tag{6}$$

由式(2)易得：

$$P\{X_2 > t_2 \mid X_1 = t_1, X_2 > t_1\} = \exp\{-\int_{t_1}^{t_2} h_2(u \mid X_1 = t_1)\mathrm{d}u\} \tag{7}$$

其中：

$$\frac{\mathrm{d}[\ln P\{X_2 > t_2 \mid X_1 = t_1, X_2 > t_1\}]}{\mathrm{d}t_2} = \frac{\mathrm{d}}{\mathrm{d}t_2}\left[\frac{\int_{t_2}^{\infty} f(t_1, x_2)\mathrm{d}x_2}{\int_{t_1}^{\infty} f(t_1, x_2)\mathrm{d}x_2}\right] \frac{\int_{t_1}^{\infty} f(t_1, x_2)\mathrm{d}x_2}{\int_{t_2}^{\infty} f(t_1, x_2)\mathrm{d}x_2} = -h_2(t_2 \mid X_1 = t_1) \tag{8}$$

式(8)两端分别关于 t_2 从 t_1 到 t_2 积分即得式(7)。类似式(7)可得：

$$R(t_1, t_1) = P\{X_1 > t_1, X_2 > t_1\} = \exp\{-\int_0^{t_1} \sum_{i=1}^{2} r_i(u \mid X_{3-i} > t_1)\mathrm{d}u\} \tag{9}$$

最后由式(6)、(7)和式(9)可得：

$$f(t_1, t_2) = r_1(t_1 \mid X_2 > t_1) h_2(t_2 \mid X_1 = t_1) \exp\{-\int_0^{t_1} \sum_{i=1}^{2} r_i(u \mid X_{3-i} > t_1)\mathrm{d}u - \int_{t_1}^{t_2} h_2(u \mid X_1 = t_1)\mathrm{d}u\} \tag{10}$$

注：该定理表明条件失效率函数能唯一确定二维随机变量的分布。

2.2　*n*维的情况

推广的条件失效率函数

$$\begin{aligned} & r_i(t_i \mid X_j > t_j, j = 1, 2, \cdots, n, j \ne i) \underset{\Delta t_i \to 0}{\triangleq \lim} \frac{P\{t_i < X_i \le t_i + \Delta t_i \mid X_i > t_i, X_j > t_j, j = 1, 2, \cdots, n, j \ne i\}}{\Delta t_i} = \\ & \frac{\int_{t_1}^{\infty} \cdots \int_{t_{i-1}}^{\infty} \int_{t_{i+1}}^{\infty} \cdots \int_{t_n}^{\infty} f(x_1, \cdots, x_{i-1}, t_i, x_{i+1}, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \cdots \mathrm{d}x_n}{P(X_1 > t_1, X_2 > t_2, \cdots, X_n > t_n)} \times \\ & \frac{f_i(t_i \mid X_j > t_j, j = 1, 2, \cdots, n, j \ne i)}{P\{X_i > t_i \mid X_j > t_j, j = 1, 2, \cdots, n, j \ne i\}} = -\frac{\mathrm{d}}{\mathrm{d}t_i} \ln P\{X_i > t_i \mid X_j > t_j, j = 1, 2, \cdots, n, j \ne i\} \end{aligned} \tag{11}$$

$$
\begin{aligned}
&\lambda_i(t_i \mid X_1=t_1,\cdots,X_{i-1}=t_{i-1},X_{i+1}>t_i,\cdots,X_n>t_i)\triangleq \\
&\lim_{\Delta t_i\to 0}\frac{P(t_i<X_i\le t_i+\Delta t_i \mid X_1=t_1,\cdots,X_{i-1}=t_{i-1},X_i>t_i,X_{i+1}>t_i,\cdots,X_n>t_i)}{\Delta t_i}= \\
&\frac{\int_{t_i}^{\infty}\cdots\int_{t_i}^{\infty} f(t_1,\cdots,t_{i-1},t_i,x_{i+1},\cdots,x_n)\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n}{\int_{t_i}^{\infty}\cdots\int_{t_i}^{\infty} f(t_1,\cdots,t_{i-1},x_i,\cdots,x_n)\mathrm{d}x_i\cdots\mathrm{d}x_n}= \\
&\frac{f_i(t_i \mid X_1=t_1,\cdots,X_{i-1}=t_{i-1},X_{i+1}>t_i,\cdots,X_n>t_i)}{P\{X_i>t_i \mid X_1=t_1,\cdots,X_{i-1}=t_{i-1},X_{i+1}>t_i,\cdots,X_n>t_i\}}= \\
&-\frac{\mathrm{d}}{\mathrm{d}t_i}\ln P\{X_i>t_i \mid X_1=t_1,\cdots,X_{i-1}=t_{i-1},X_{i+1}>t_i,\cdots,X_n>t_i\}
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&h_i(t_i \mid X_j=t_j,j=1,2,\cdots,n,j\ne i)\triangleq\lim_{\Delta t_i\to 0}\frac{P\{t_i<X_i\le t_i+\Delta t_i \mid X_i>t_i,X_j=t_j,j=1,2,\cdots,n,j\ne i\}}{\Delta t_i}= \\
&\frac{f(t_1,t_2,\cdots,t_n)}{\int_{t_i}^{\infty} f(t_1,t_2,\cdots,t_{i-1},x_i,t_{i+1},\cdots,t_n)\mathrm{d}x_i}= \\
&\frac{f_i(t_i \mid X_j=t_j,j=1,2,\cdots,n,j\ne i)}{P\{X_i>t_i \mid X_j=t_j,j=1,2,\cdots,n,j\ne i\}}= \\
&-\frac{\mathrm{d}}{\mathrm{d}t_i}\ln P\{X_i>t_i \mid X_j=t_j,j=1,2,\cdots,n,j\ne i\}
\end{aligned}
\tag{13}
$$

以上均有$i,j=1,2,\cdots,n$。

推论 1 如果f是非负随机变量$(X_1,X_2,\cdots,X_n)$的联合概率密度函数，那么有：

$$
\begin{aligned}
&f(t_1,t_2,\cdots,t_n)=r_{i_1}(t_{i_1} \mid X_{i_2}>t_{i_1},\cdots,X_{i_n}>t_{i_1})\prod_{j=2}^{n-1}[\lambda_{i_j}(t_{i_j} \mid X_{i_1}=t_{i_1},\cdots,X_{i_{j-1}}=t_{i_{j-1}},X_{i_{j+1}}>t_{i_j},\cdots,X_{i_n}>t_{i_j})]\times \\
&h_{i_n}(t_{i_n} \mid X_{i_1}=t_{i_1},\cdots,X_{i_{n-1}}=t_{i_{n-1}})\exp\{-[\int_0^{t_{i_1}}\sum_{j=1}^{n} r_{i_j}(u \mid X_{i_k}>t_{i_1},j\ne k,k=1,2,\cdots,n)+ \\
&\sum_{j=2}^{n-1}\sum_{k=j}^{n}\int_{t_{i_{j-1}}}^{t_{i_j}}\lambda_{i_k}(u \mid X_{i_1}=t_{i_1},\cdots,X_{i_{j-1}}=t_{i_{j-1}},X_{i_j}>t_{i_j},\cdots,X_{i_{k-1}}>t_{i_j},X_{i_{k+1}}>t_{i_{j-1}},\cdots,X_{i_n}>t_{i_{j-1}})+ \\
&\int_{t_{i_{n-1}}}^{t_{i_n}} h_{i_n}(u \mid X_{i_1}=t_{i_1},\cdots,X_{i_{n-1}}=t_{i_{n-1}})]\mathrm{d}u\}
\end{aligned}
\tag{14}
$$

式中 随机变量$X_{i_1},X_{i2},\cdots,X_{i_n}$的观察值依次为$t_{i_1},t_{i2},\cdots,t_{i_n}$，$(i_1,i_2,\cdots,i_n)$是$(1,2,\cdots,n)$满足$0\le t_{i_1}\le t_{i2}\le\cdots\le t_{i_n}$的某个排列。

参 考 文 献

[1] 曹晋华, 程 凯. 可靠性数学引论[M]. 北京: 科学出版社, 1986

[2] Cox D R. Regression models and lifetables (with discussion)[J]. J. Roy. Statist. Soc.Ser., 1972, 34(6): 187-202

[3] Shaked M, Shanthikumar J G. Multivariate imperfect repair[J]. Operations Research, 1986, 34 (3): 437-448

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