Eigenvalue Theorem on Near-Algebra and Banach Algebra

YANG Han-sheng[1], ZHONG Shou-ming[2]

(1. Dep. of Mathematics, Southwest University of Science and Technology Sichuan Mianyang 621002;

2. School of Applied Mathematics, UEST of China Chengdu 610054)

Abstract Let a (p, q)-additive selfmap f on near-algebra or Banach algebra X satisfy $f(e)=e$ and $f(u)=\phi(u)f(u^{-1})\varphi(u)$ where $\phi: X\to X$ and $\varphi: X\to DES(X)$ be an automorphism and antiautomorphism respectively such that $\phi(u)=u\varphi(u^{-1})u$ for each invertible u of X. Then all of the normal invertibles of X have the common eigenvalue $\lambda=2q/(1+q)$ if $p=q\neq-1$.

Key words automorphism; antiautomorphism; normal invertible; eigenvalue

NEAR-ALGEBRA和BANACH代数上的一个特征值定理

杨汉生[1]，钟守铭[2]

(1. 西南科技大学理学院 四川 绵阳 621002; 2. 电子科技大学应用数学学院 成都 610054)

【摘要】在near-algebra或 Banach代数中引入(p, q)-可加自映象 f 和正则可逆元的概念，得到一个值得注意的结果，即在一定条件下，对于定义在near-algebra或 Banach代数X中(p, q)-可加自映象f，X中的任意正则可逆元都具有公共的特征值 $\lambda=2q/(1+q)$， $p=q\neq-1$。

关 键 词 自同构; 反自同构; 正则可逆; 特征值

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The Refs.[1-6] show that the study to *near-ring* is surpassed the close range of pure abstract algebra. In this paper we introduce the concept of *near-algebra*, and try to explore a special way to research some fixed-point and eigenvalue questions in range of functional analysis by means of abstract algebraic method[7].

Ref.[8] studied derivations in prime rings and Banach algebras and some functional equation in Banach algebras[7], and Ref.[9] had discussion for derivations in prime near-rings. Ref.[10] researched fixed points of the mapping class group in the $SU(n)$ moduli spaces. Now we are interested in the following questions:

(1) Which mapping has fixed point in near-algebras and Banach algebras?

(2) What mapping has eigenvalue in near-algebras and Banach algebras?

It is high time that the study to fixed points and eigenvalue should be introduced into near-algebras and Banach algebras. This paper answers the questions above.

Definition 1 A set X, together with two binary operations + and $\bullet$, is called a (left) near-ring, denoted (X,

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作者简介：杨汉生(1947 –)，男，教授，主要从事非线性分析与基础数学方面的研究.

+,•), if:

(1) (*X*, +) is a group (not necessarily abelian);

(2) (*X*, •) is a semigroup;

(3) $x(y+z)=xy+yz$ for all $x,y,z\in X$.

An element $d\in X$ is called distributive if $(x+y)d=xd+yd$ for all $x,y\in X$. The subset *DES* (*X*) of the distributive elements in X forms a subsemigroup of (*X*, •)[10].

A (left) near-ring (*X*, +,•) with identity *e* over scalars *K* is called a (left) *near-algebra* over *K*, denoted (*X*, *K*, +,•), if for any *x* in *X* and λ in *K* a product $\lambda x\in X$ is defined in such a way that the following rule holds: $\lambda(xy)=(\lambda x)y=x(\lambda y)$, $y\in X$. Clearly *Banach algebra* is a special case of near-algebra.

Definition 2 A mapping $f:X\to X$ is called (p,q)-additive if there are p , $q\in K$, $q\neq 0$, such that $f(e\pm u)=pf(e)\pm qf(u)$ for any invertible element *u* in *X*, where *X* is a near-algebra, with identity *e*, over scalars K .

Definition 3 An invertible element *u* in *X* is called to be normal if $(e-u)^{-1}$ existing in *X*.

Theorem Let (*X*, *K*, +,•) be a left near-algebra, with identity *e*, over scalars *K*. Let a (*p*, *q*)-additive selfmap *f* of *X* satisfy $f(e)=e$ and $f(u)=\phi(u)f(u^{-1})\varphi(u)$ where $\phi:X\to X$ and $\varphi:X\to DES(X)$ be an automorphism and antiautomorphism (i.e., $\varphi(uv)=\varphi(v)\varphi(u)$) respectively such that $\phi(u)=u\varphi(u^{-1})u$ for each invertible *u* of *X*. Then all of the normal invertibles of *X* have the common eigenvalue, of map *f*, $\lambda=2q/(1+q)$ where $p=q\neq -1$.

Proof Obviously, *N* = {all of the normal invertibles of *X*} $\neq \varPhi$, for example, $e/2\in N$. Let

$$g(u)=f(u)-u \tag{1}$$

then

$$g(e)=f(e)-e=0 \tag{2}$$

Now we note that *f* is (p,q)-additive and

$$e=(e-u)^{-1}(e-u)=(e-u)^{-1}-(e-u)^{-1}u \tag{3}$$

hence

$$g(e\pm u)=pf(e)\pm qf(u)-(e\pm u)=\pm qg(u)+(p-1)e\pm(q-1)u \tag{4}$$

$$g((e-u)^{-1})=g(e+(e-u)^{-1}u)=qg((e-u)^{-1}u)+(p-1)e+(q-1)(e-u)^{-1}u \tag{5}$$

$$g(u^{-1})=g(e+u^{-1}(e-u))=qg(u^{-1}(e-u))+(p-q)e+(q-1)u^{-1} \tag{6}$$

On the other side, from $\varphi(u)\in \mathrm{DES}(X)$ and other conditions of the theorem, we obtain

$$\phi(u)g(u^{-1})\varphi(u)=\phi(u)f(u^{-1})\varphi(u)-\phi(u)u^{-1}\varphi(u)=f(u)-u\varphi(u^{-1})uu^{-1}\varphi(u)=g(u) \tag{7}$$

Now we have

$$\begin{aligned}g(u)=\phi(u)g(u^{-1})\varphi(u)=\phi(u)[qg(u^{-1}(e-u))+(p-q)e+(q-1)u^{-1}]\varphi(u)=\\ q\phi(u)g(u^{-1}(e-u))\varphi(u)+(p-q)\phi(u)\varphi(u)+(q-1)u\varphi(u^{-1})uu^{-1}\varphi(u)=\\ q\phi(u)\phi(u^{-1}(e-u))g((e-u)^{-1}u)\varphi(u^{-1}(e-u))\varphi(u)+(p-q)\phi(u)\phi(u)+(q-1)u=\\ q\phi(uu^{-1}(e-u))g((e-u)^{-1}u)\varphi(uu^{-1}(e-u))+(p-q)\phi(u)\phi(u)+(q-1)u=\\ q\phi(e-u)g((e-u)^{-1}u)\varphi(e-u)+(p-q)\phi(u)\phi(u)+(q-1)u=\\ \phi(e-u)[g((e-u)^{-1})+(1-p)e+(1-q)(e-u)^{-1}u]\varphi(e-u)+(p-q)\phi(u)\phi(u)+(q-1)u=\\ \phi(e-u)\phi((e-u)^{-1})g(e-u)\varphi((e-u)^{-1})\varphi(e-u)+r(u)=\\ \phi((e-u)(e-u)^{-1})g(e-u)\varphi((e-u)(e-u)^{-1})+r(u)=\\ -qg(u)+(p-1)e+(1-q)u+2(q-1)u+(1-p)e+(p-q)(\phi(u)+\varphi(u))=\\ -qg(u)+(q-1)u+(p-q)(\phi(u)+\varphi(u))\end{aligned}$$

where

$$r(u)=(1-p)\phi(e-u)\varphi(e-u)+(1-q)\phi(e-u)(e-u)^{-1}u\varphi(e-u)+(p-q)\phi(u)\varphi(u)+(q-1)u=$$
$$(1-p)\phi(e-u)\varphi(e-u)+(1-q)\phi(e-u)[(e-a)^{-1}-e]\varphi(e-u)+(p-q)\phi(u)\varphi(u)+(q-1)u=$$
$$(1-p)\phi(e-u)\varphi(e-u)+(1-q)[\phi(e-u)(e-a)^{-1}\varphi(e-u)-\phi(e-u)\varphi(e-u)]+(p-q)\phi(u)\varphi(u)+(q-1)u=$$
$$(1-q)(e-u)\varphi((e-u)^{-1})(e-a)(e-a)^{-1}\varphi(e-u)+(q-1)u+(q-p)[\phi(e-u)\varphi(e-u)-\phi(u)\varphi(u)]=$$
$$(1-q)(e-u)\varphi((e-u)(e-u)^{-1})+(q-p)[(e-\phi(u))(e-\varphi(u))-\phi(u)\varphi(u)]+(q-1)u=$$
$$(1-q)(e-u)+(q-p)[e-\phi(u)-\varphi(u)+\phi(u)\varphi(u)-\phi(u)\varphi(u)]+(q-1)u=$$
$$(1-p)e+2(q-1)u+(p-q)(\phi(u)+\varphi(u))$$

Thus

$$(1+q)g(u)=(p-q)(\phi(u)+\varphi(u))-(1-q)u \tag{8}$$

Obviously, Eq.(8) yields the following conclusions:

1) $f(u)=\lambda u$, $\lambda=2q/(1+q)$, if $p=q\neq -1$;

2) $f(u)=u$, if $(p-q)\ (\phi(u)+\varphi(u))=(1-q)u$ and $q\neq -1$;

3) $f(u)=u$, if $p=q=1$;

4) $\phi(u)+\varphi(u)=\lambda u$, $\lambda=2/(p+1)$, if $p\neq q=-1$.

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·科研成果介绍·

CCD挠度测量系统

CCD挠度测量系统用于实时监测大型桁架结构的挠度。利用激光束的直线性作为挠度测量的基准，激光束照射在CCD器件上的光点位置的变化即反映挠度变化。激光源和CCD接收单元的空间位置关系可通过操作界面进行预置，通过光点位置可直接计算桁架结构的变形状况。本系统还有存储、查看、分析历史记录的功能。

其主要技术指标：(1) 挠度的测量精度：±0.05 mm；(2) 挠度的测量范围：10 mm；(3) 拥有8个测量通道；总数据采集时间共计不大于0.5 s。

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