· 物理电子学 ·

Strong Convergence Theorem for κ-Strict Pseudo-Contractions in Hilbert Spaces

YANG Li

(Department of Mathematics and Physics, Southwest University of Science and Technology　Mianyang Sichuan　621002)

Abstract In an infinite dimensional Hilbert space, the normal Mann's iterative algorithm has only weak convergence, in general, even for non-expansive mappings. In order to get a strong convergence result, the normal Mann's iterative algorithm is modified by using a suitable convex combination of a fixed vector and a sequence in a closed convex subset of a real Hilbert space. A strong convergence theorem is established by means of a new Ishikawa-like iterative algorithm for κ-strict pseudo-contractions in Hilbert spaces. The results presented in this paper have extended and improved some recent results .

Key words convergence theorem; iterative algorithm; metric projection; non-expansive mappings; κ-strict pseudo-contraction

Hilbert空间中κ-严格伪压缩的强收敛定理

杨　莉

(西南科技大学理学院　四川 绵阳　621002)

【摘要】 在无穷维 Hilbert 空间中，即使对非扩张映像 Mann，迭代算法仅有弱收敛。为了得到强收敛定理，该文利用 Hilbert 空间中闭凸子集的一个序列和一个给定向量作适当的凸组合修改 Mann 迭代算法，在 Hilbert 空间中给出了一个新的 κ-严格伪压缩修正的 Mann 迭代算法——似 Ishikawa 迭代算法，并且建立了该算法的强收敛定理。推广和改进了一些最新的结果。

关键词　收敛定理; 迭代算法; 距离投影; 非扩张映像; κ-严格伪压缩
中图分类号　O177.91　　**文献标识码**　A　　doi:10.3969/j.issn.1001-0548.2009.04.017

1　Introduction and Preliminaries

The normal Mann's iterative algorithm generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n \geqslant 1 \qquad (1)$$

where $\{\alpha_n\}$ is a real sequence in (0,1) which satisfies certain control conditions.

In an infinite dimensional Hilbert space, the normal Mann's iterative algorithm has only weak convergence. In order to get a strong convergence result, one has to modify the normal Mann's iterative algorithm. Some attempts have been made and several important results have been reported [1-2,4-12].

The purpose of this paper is to establish a strong convergence theorem by means of the Ishikawa-like algorithm. The results presented in this paper are improvements and extensions of the corresponding results in Refs.[4-6].

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, C is a nonempty closed convex subset of H and T: $C \to H$ is a mapping. We use $F(T)$ to denote the fixed point set of T and P_C to denote the metric projection of H onto C. $\to$ and $\xrightarrow{w}$ denote strong and weak convergence, respectively.

Recall that T: $C \to H$ is called a κ-strict pseudo-contraction if there exists a constant $\kappa \in [0,1)$ such that:

Received date: 2008－03－05;　Revised date: 2008－09－17
收稿日期：2008－03－05；修回日期：2008－09－17
Foundation item: Supported by the Natural Science Foundation of Sichuan Province (08ZA008)
基金项目：四川省自然科学基金(08ZA008)
Biography: Yang li was born in 1962. She is an associate professor, her research interest is in the nonlinear analysis.
作者简介：杨　莉(1962－)，女，副教授，主要从事非线性分析方面的研究.

$$\|Tx-Ty\|^2 \leqslant \|x-y\|^2+\kappa\|(I-T)x-(I-T)y\|^2 \qquad (2)$$

for all $x, y \in C$.

When $\kappa=0$, T is said to be non-expansive, and it is said to be pseudo-contraction if $\kappa=1$.

In order to establish the convergence theorem, we collect some facts.

Lemma 1[1] Let H be a real Hilbert space. There hold the following identities:

(1) $\|x \pm y\|^2=\|x\|^2 \pm 2\langle x, y\rangle+\|y\|^2, \forall x, y \in H$;

(2) $\forall t \in[0,1], \forall x, y \in H: \|tx+(1-t)y\|^2=t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2$

Lemma 2[1] Let K be a nonempty closed convex subset of a real Hilbert space H. For every $x \in H$, there exists a unique point $z \in K$, such that $\|x-z\| \leqslant \|x-y\|$ for all $y \in K$, Define $P_K: H \rightarrow K$ by $z=P_K x$. Then $z=P_K x$ if and only if $\langle x-z, y-z\rangle \leqslant 0$, for all $y \in K$.

Lemma 3[2-3] Let $\{a_n\},\{b_n\}$ and $\{c_n\}$ be three real sequences satisfying the following condition:

$$a_{n+1} \leqslant (1-\omega_n) a_n+b_n+c_n \quad \forall n \geqslant n_0$$

where n_0 is some nonnegative integer, $\{\omega_n\}$ is a sequence in (0,1) with $\sum_{n=1}^{\infty} \omega_n=\infty$ if one of the following conditions hold:

(1) $\{a_n\},\{b_n\}$ and $\{c_n\}$ are all nonnegative real sequences and $b_n=\circ(\omega_n), \sum_{n=1}^{\infty} c_n<\infty$.

(2) $\{a_n\}$ is nonnegative real sequence, $b_n=0, n \in N$ and $\{c_n\}$ is a real sequence such that $\limsup_{n \rightarrow \infty} \frac{c_n}{\omega_n} \leqslant 0$, Then $\lim_{n \rightarrow \infty} a_n=0$.

Lemma 4[4] If T is a κ-strict pseudo-contraction on closed convex subset C of a real Hilbert space H, then $I-T$ is demiclosed at any point $y \in H$.

Lemma 5[4] If T: $C \rightarrow H$ is a κ-strict pseudo-contraction, then T is Lipschitzian.

Lemma 6[4] If T is a κ-strict pseudo-contraction on a closed convex subset C of a real Hilbert space H, then the fixed point set $F(T)$ of T is closed convex so that the projection $P_{F(T)}$ is well defined.

Lemma 7[4] Let T: $C \rightarrow H$ be a κ-strict pseudo-contraction with $F(T) \neq \phi$. Then $F(P_C T)=F(T)$.

Lemma 8[4] Let T: $C \rightarrow H$ be a κ-strict pseudo-contraction. Define S: $C \rightarrow H$ by $Sx=\lambda x+(1-\lambda) Tx$ for each $x \in C$. Then, as $\lambda \in[\kappa, 1)$, S is non-expansive such that $F(S)=F(T)$.

2 Main results

Below is a modification of the normal Mann's iterative algorithm for κ-strict pseudo-contractions.

Theorem 1 Let C be a nonempty closed convex subset of a real Hilbert space H, $T: C \rightarrow H$ be a κ-strict pseudo-contraction non-self mapping such that $F(T) \neq \phi$ and f: $H \rightarrow H$ be a contraction mapping with a contraction constant $\xi \in(0,1)$, sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ and $\{\lambda_n\}$ in (0,1), the following control conditions are satisfied:

(I) $\alpha_n+\beta_n+\gamma_n=1$,

(II) $\alpha_n \rightarrow 0\ (n \rightarrow \infty), \sum_{n=1}^{\infty} \alpha_n=\infty, \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$,

(III) $\beta_n \rightarrow 0\ (n \rightarrow \infty), \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$,

(IV) $\kappa \leqslant \lambda_n \leqslant b<1$, for all $n \geqslant 1$

$$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$$

Let a sequence $\{x_n\}$ be generated in the following manner:

$$\begin{cases} x_1 \in C \\ y_n=P_C[\lambda_n x_n+(1-\lambda_n) T x_n] \\ x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n y_n \end{cases} \qquad (3)$$

Then $\{x_n\}$ converges strongly to a fixed point z of T, where $z=P_{F(T)} f(z)$.

Proof Lemma 6 ensures that $P_{F(T)}$ is well defined. Since f: $H \rightarrow H$ be a contraction mapping with a contraction constant ξ, hence:

$$\|P_{F(T)} f(x)-P_{F(T)} f(y)\| \leqslant \xi\|x-y\| \quad \forall x, y \in H$$

By Banach theorem, there exists a unique $z \in F(T)$ such that $z=P_{F(T)} f(z)$.

By using Lemma 1, (2), (3) and condition (IV), we have, for $p \in F(T)$:

$$\begin{aligned} \|y_n-p\|^2 &\leqslant \|\lambda_n x_n+(1-\lambda_n) T x_n-p\|^2= \\ &\|\lambda_n(x_n-p)+(1-\lambda_n)(T x_n-p)\|^2= \\ &\lambda_n\|x_n-p\|^2+(1-\lambda_n)\|T x_n-p\|^2- \\ &\lambda_n(1-\lambda_n)\|x_n-T x_n\|^2=\|x_n-p\|^2-(\lambda_n-\kappa)(1-\lambda_n) \\ &\|x_n-T x_n\|^2 \leqslant \|x_n-p\|^2 \end{aligned} \qquad (4)$$

By using (3), (4), condition (I) and induction, we obtain:

$$\|x_n - p\| = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\| \leqslant$$
$$\alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \leqslant$$
$$\alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (\beta_n + \gamma_n)\|x_n - p\| \leqslant$$
$$\alpha_n \xi \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| =$$
$$(1-(1-\xi)\alpha_n)\|x_n - p\| + \alpha_n (1-\xi)\frac{1}{1-\xi}\|f(p) - p\| \leqslant$$
$$\max\left\{\|x_1 - p\|, \frac{1}{1-\xi}\|f(p) - p\|\right\} = M_1$$

Consequently, both $\{x_n\}$ and $\{y_n\}$ are bounded. Since f is a contraction mapping, by Lemma 5 T is Lipschitzian, so $\{f(x_n)\}$ and $\{T(x_n)\}$ are both bounded.

Define $T_n x = P_C[\lambda_n x + (1-\lambda_n)Tx]$, then $T_n : C \to C$ is non-expansive.

Next we prove that $x_{n+1} - x_n \to 0$ as $n \to \infty$.

To this end, we first estimate $\|y_n - y_{n-1}\|$. Set:

$$M_2 = \max\{\sup\{\|x_n\|\}, \sup\{\|y_n\|\}, \sup\{\|f(x_n)\|\},$$
$$\sup\{\|T(x_n)\|\}\}$$

Then by using Equ.(3) and noting that T_n is non-expansive, we have:

$$\|y_n - y_{n-1}\| = \|T_n x_n - T_{n-1}x_{n-1}\| =$$
$$\|T_n x_n - T_n x_{n-1} + T_n x_{n-1} - T_{n-1}x_{n-1}\| =$$
$$\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1}x_{n-1}\| \leqslant$$
$$\|x_n - x_{n-1}\| + 2M_2|\lambda_n - \lambda_{n-1}| \qquad (5)$$

At this point, from Equ. (3) and (5), we get:

$$\|x_{n+1} - x_n\| \leqslant \alpha_n \|f(x_n) - f(x_{n-1})\| +$$
$$|\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|y_{n-1}\|) + \beta_n \|x_n - x_{n-1}\| +$$
$$\gamma_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|x_{n-1}\| + \|y_{n-1}\|) \leqslant$$
$$\alpha_n \xi \|x_n - x_{n-1}\| + 2M_2(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) +$$
$$(1-\alpha_n)\|x_n - x_{n-1}\| + 2M_2\gamma_n|\lambda_n - \lambda_{n-1}| \leqslant$$
$$(1-(1-\xi)\alpha_n)\|x_n - x_{n-1}\| +$$
$$2M_2(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\lambda_n - \lambda_{n-1}|) \qquad (6)$$

By using Lemma 3, we conclude that $x_{n+1} - x_n \to 0$ as $n \to \infty$. Hence it follows from Equ. (5) that $y_{n+1} - y_n \to 0$ as $n \to \infty$.

Noting that condition (II), (III), we obtain:

$$\|x_{n+1} - y_{n+1}\| = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - y_{n+1}\| =$$
$$\alpha_n \|f(x_n) - y_{n+1}\| + \beta_n \|x_n - y_{n+1}\| + \gamma_n \|y_n - y_{n+1}\| =$$
$$2M_2(\alpha_n + \beta_n) + \gamma_n \|y_{n+1} - y_n\| \to 0 \qquad n \to \infty$$

On the other hand, by condition (IV), we have $\lambda_n \to \lambda$ as $n \to \infty$, where $\lambda \in [\kappa, 1)$.

Define $S : C \to H$, by $Sx = \lambda x + (1-\lambda)Tx$. Then S is non-expansive mapping with $F(S) = F(T)$ by Lemma 8. It follows from Lemma 7 that:

$$F(P_C S) = F(S) = F(T)$$

Since $\|P_C S x_n - y_n\| \leqslant 2M_2|\lambda_n - \lambda| \to 0$ as $n \to \infty$, we have:

$$\|x_n - P_C S x_n\| \leqslant \|x_n - y_n\| + \|P_C S x_n - y_n\| \to 0 \quad n \to \infty$$

We now prove that $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leqslant 0$, where $z = P_{F(T)} f(z)$. To see this, assume that:

$$\limsup_{n\to\infty}\langle f(z) - z, y_n - z\rangle = \lim_{j\to\infty}\langle f(z) - z, y_{n_j} - z\rangle$$

Without loss of generality, assume that $y_{n_j} \xrightarrow{w} p$ as $j \to \infty$, then $x_{n_j} \xrightarrow{w} p$ and $x_{n_j} - P_C S x_{n_j} \to 0$ as $j \to \infty$. By virtue of Lemma 4, we have $p \in F(P_C S) = F(T)$. By Lemma 2, we know that $\langle f(z) - z, p - z\rangle \leqslant 0$. Hence:

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle =$$
$$\limsup_{n\to\infty}\langle f(z) - z, x_n - y_n + y_n - z\rangle =$$
$$\limsup_{n\to\infty}\langle f(z) - z, y_n - z\rangle \leqslant 0$$

Finally, we prove that $x_n \to z$ as $n \to \infty$.

By using Lemma 1, (3), and (4), we have:

$$\|x_{n+1} - z\|^2 = \langle \alpha_n (f(x_n) - z) + \beta_n (x_n - z) +$$
$$\gamma_n (y_n - z), x_{n+1} - z\rangle \leqslant \alpha_n \langle f(x_n) - f(z), x_{n+1} - z\rangle +$$
$$\alpha_n \langle f(z) - z, x_{n+1} - z\rangle + \beta_n \|x_n - z\|\|x_{n+1} - z\| +$$
$$\gamma_n \|y_n - z\|\|x_{n+1} - z\| \leqslant \frac{1}{2}\alpha_n \xi(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) +$$
$$\alpha_n \langle f(z) - z, x_{n+1} - z\rangle + \frac{1}{2}\beta_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) +$$
$$\frac{1}{2}\gamma_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) = \frac{1}{2}[1-(1-\xi)\alpha_n]\times$$
$$\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle$$

which implies that $\|x_{n+1} - z\|^2 \leqslant [1-(1-\xi)\alpha_n]\times \|x_n - z\|^2 + 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle$ By virtue of Lemma 3 $x_n \to z$ as $n \to \infty$.

The theorem extends main results of Refs.[4-6], Putting $f \equiv 0$, we obtain Theorem 1 of Ref.[4], again putting $\kappa = 0$, we obtain Theorem 1 of Ref. [5], putting $\kappa = 0$ and $\alpha_n \equiv 0$, we obtain the known Wittmann's convergence theorem[6]. Our theorem is also related to those results of Ref. [12], but they are different convergence theorems.

(下转第563页)