一类具比例时滞脉冲递归神经网络的全局多项式稳定性

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【摘要】该文对一类具比例时滞脉冲递归神经网络给出全局多项式稳定性定义。通过引入可调参数，构造适合的 Lyapunov 泛函和运用线性矩阵不等式 (LMI) 的方法，对该系统的全局多项式稳定性进行讨论，得到了保证该系统全局多项式稳定的判定准则，且这些准则以 LMI 的形式给出的，方便应用 Matlab 工具箱进行验证。该文还揭示了多项式稳定性与指数稳定性之间的关系，最后通过数值算例验证了所得准则的准确性。

关 键 词　脉冲效应；　Lyapunov 泛函；　比例时滞；　多项式稳定性；　递归神经网络

中图分类号　TP183；O175.13　　**文献标志码**　A　　doi:10.12178/1001-0548.2019171

Global Polynomial Stability of a Class of Impulsive Recurrent Neural Networks with Proportional Delays

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Abstract　The definition of global polynomial stability for a class of impulsive recurrent neural networks (IRNNs) with proportional delays is given. By introducing adjustable parameters, several suitable Lyapunov functionals are constructed and the method of linear matrix inequality (LMI) is used to discuss the global polynomial stability of the system. Several criteria for guaranteeing the global polynomial stability of the system are obtained. And these criteria are given in the form of LMI, which is convenient to use Matlab toolbox for verification. The relationship between polynomial stability and exponential stability is revealed. The criteria are verified by numerical examples.

Key words　impulse effect;　Lyapunov functional;　proportional delays;　polynomial stability;　recurrent neural networks

时滞递归神经网络 (DRNNs) 基于在联想记忆、优化控制、图像处理等领域的应用而被广泛研究。在这些应用中大多数都需要 DRNNs 是稳定的，指数稳定性作为 DRNNs 的一种重要的动力学性质[1-4]，它的特征之一是 Lyapunov 指数不等于 0。然而大多数情况下，DRNNs 的最大 Lyapunov 指数是等于零的，其状态轨迹渐近趋于平衡点，即此时 DRNNs 是渐近稳定的[5-7]。

多项式稳定性是一种特殊的稳定性，它同指数稳定性一样蕴含着渐近稳定性，但其收敛速度比指数稳定性慢一些。它的特征之一就是其最大的 Lyapunov 指数等于零。目前，这种系统的多项式稳定性研究较少，只有某些系统在某些特殊情况下才具有多项式稳定性，如波动方程[8-9] 和随机微分方程[10-14]。需要说明的是，这里的多项式稳定性不是指对于一个多项式来研究这个多项式的稳定性，而是因为某些系统的解的估计式中含有 $t^{-\lambda}$ $(t \geqslant t_0, \lambda > 0)$，类似于多项式，故称这种稳定性为多项式稳定性，具体见下文的定义。

由上所述，某种 DRNNs 是否具有多项式稳定性？文献 [15] 回答了这个问题，文中给出了比例时滞递归神经网络 (RNNs) 多项式稳定性的定义，并研究了几类比例时滞 RNNs 的多项式稳定性和多项式周期性。文献 [16] 将比例时滞引入细胞神经网络，提出比例时滞细胞神经网络模型。此后各种比例时滞 RNNs 基于在二次规划问题和 QoS 路由

收稿日期：2019－08－16；修回日期：2020－09－21
基金项目：天津市自然科学基金 (18JCYBJC85800)；天津市高校中青年骨干创新培养计划 (135205GC38)
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决策等方面的潜在应用得到了国内外学者的关注，并取得一些研究成果[17-26]，但关于比例时滞 RNNs 的多项式稳定性的研究还很有限。

除了时滞效应外，脉冲也是影响神经网络动力学性质的重要因素之一。脉冲效应是指在网络运行过程中，系统的状态发生突变，导致网络的行为更加复杂，因此研究脉冲作用下系统的稳定性是非常必要的。目前，已有许多关于脉冲 DRNNs 的动力学行为的研究成果[24, 27-32]。文献 [24] 应用 Lyapunov 稳定性理论结合线性矩阵不等式的方法研究一类比例时滞脉冲 RNNs (IRNNs) 的无源性。然而，比例时滞 IRNNs 的动力学行为的研究还很少。基于此，本文通过构造 Lyapunov 泛函和 LMI 的方法对一类比例时滞 IRNNs的全局多项式稳定性进行探讨。

1 模型与预备知识

设$C([\tilde{q}t_0,t_0],\mathbb{R}^n)$和$C([\log\tilde{q}t_0,\log t_0],\mathbb{R}^n)$分别表示具一致收敛拓扑的从$[\tilde{q}t_0,t_0]\to\mathbb{R}^n$和$[\log\tilde{q}t_0,\log t_0]\to\mathbb{R}^n$的所有连续函数构成的 Banach 空间，这里$t_0\geqslant 1$。对于$\boldsymbol{\varphi}\in C([\tilde{q}t_0,t_0],\mathbb{R}^n)$，$\|\boldsymbol{\varphi}\|_{\tilde{q}}^2=\sup_{\tilde{q}t_0\leqslant s\leqslant t_0}\|\boldsymbol{\varphi}(s)\|^2$，$\boldsymbol{\eta}\in C([\log\tilde{q}t_0,\log t_0],\mathbb{R}^n)$，$\|\boldsymbol{\eta}(s)\|_{\tilde{\tau}}^2=\sup_{\log\tilde{q}t_0\leqslant s\leqslant\log t_0}\|\boldsymbol{\eta}(s)\|^2$。对于$\boldsymbol{A}\in\mathbb{R}^{n\times n}$，$\boldsymbol{A}>0$和$\boldsymbol{A}<0$分别$\boldsymbol{A}$为正定和负定，$\lambda_M(\boldsymbol{A})$和$\lambda_m(\boldsymbol{A})$分别表示$\boldsymbol{A}$的最大和最小特征值。对于 $\boldsymbol{x}=(x_1,x_2,\cdots x_n)^{\mathrm{T}}\in\mathbb{R}^n$，$\|\boldsymbol{x}\|=\left(\sum_{i=1}^{n}x_i^2\right)^{1/2}$，$\mathbb{N}^*=\{1,2,\cdots\}$。

考虑如下脉冲 RNNs：

$$\begin{cases}\dot{\boldsymbol{x}}(t)=-\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(t))+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{x}(qt))+\boldsymbol{u}\\ \qquad t\geqslant t_0,\ t\neq t_k\\ \Delta\boldsymbol{x}(t_k)=\boldsymbol{E}_k(\boldsymbol{x}(t_k^-))\qquad k\in\mathbb{N}^*\\ \boldsymbol{x}(s)=\boldsymbol{\varphi}(s)\qquad s\in[\tilde{q}t_0,t_0]\end{cases}\tag{1}$$

式中，$\boldsymbol{x}(t)=(x_1(t),x_2(t),\cdots,x_n(t))^{\mathrm{T}}$为神经元的状态向量，$n$为神经元的个数；$\boldsymbol{C}=\mathrm{diag}(c_1,c_2,\cdots,c_n)^{\mathrm{T}}$，$c_i>0$；矩阵$\boldsymbol{A}=(a_{ij})_{n\times n}$和$\boldsymbol{B}=(b_{ij})_{n\times n}$分别表示反馈和延时反馈矩阵；$\boldsymbol{f}(\boldsymbol{x}(t))=(f_1(x_1(t)),f_2(x_2(t)),\cdots,f_n(x_n(t)))^{\mathrm{T}}$表示激活向量函数，$\boldsymbol{f}(\boldsymbol{x}(qt))=(f_1(x_1(q_1t)),f_2(x_2(q_2t)),\cdots,f_n(x_n(q_nt)))^{\mathrm{T}}$；$q_it=t-(1-q_i)t$，这里$q_i\in(0,1)$，$i=1,2,\cdots,n$。当$t\to+\infty$时，$(1-q_i)t\to+\infty$是无界时滞函数，$\tilde{q}=\min\limits_{1\leqslant i\leqslant n}\{q_i\}$；$\boldsymbol{u}=(u_1,u_2,\cdots,u_n)^{\mathrm{T}}$是外部输入常向量；$\boldsymbol{\varphi}(s)\in C([\tilde{q}t_0,t_0],\mathbb{R}^n)$表示$\boldsymbol{x}(s)(s\in[\tilde{q}t_0,t_0])$的初值函数；时间序列$\{t_k\}$满足$t_0<t_1<\cdots<t_k<t_{k+1}<\cdots$，且 $\lim_{k\to+\infty}t_k=+\infty$；$\Delta\boldsymbol{x}(t_k)=(\Delta x_1(t_k),\Delta x_2(t_k),\cdots,\Delta x_n(t_k))^{\mathrm{T}}$，$\Delta x_i(t_k)=x_i(t_k^+)-x_i(t_k^-)$，$x_i(t_k^+)=x_i(t_k)$，且$x_i(t_k^-)=\lim_{t\to t_k^-}x_i(t)$；$\boldsymbol{E}_k(\cdot)$表示状态变量在时刻$t_k$处的增量变化。

本文对激活函数作如下假设：

(H) 假设$f_i(\cdot)$ $(i=1,2,\cdots,n)$满足：

$$0\leqslant\frac{f_i(u)-f_i(v)}{u-v}\leqslant l_i\qquad u,v\in\mathbb{R}$$

记$\boldsymbol{L}=\mathrm{diag}(l_1,l_2,\cdots,l_n)$。

令$\boldsymbol{z}(t)=\boldsymbol{x}(\mathrm{e}^t)$，则式 (1) 等价地变换为：

$$\begin{cases}\dot{\boldsymbol{z}}(t)=\mathrm{e}^t[-\boldsymbol{C}\boldsymbol{z}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{z}(t))+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{z}(t-\tau))+\boldsymbol{u}]\\ \qquad t\geqslant\log t_0,\ t\neq t_k\\ \Delta\boldsymbol{z}(t_k)=\boldsymbol{E}_k(\boldsymbol{z}(t_k^-))\qquad k\in\mathbb{N}^*\\ \boldsymbol{z}(s)=\boldsymbol{\psi}(s)\qquad s\in[\log\tilde{q}t_0,\log t_0]\end{cases}\tag{2}$$

式中，$\boldsymbol{z}(t)=(z_1(t),z_2(t),\cdots,z_n(t))^{\mathrm{T}}$；$\boldsymbol{f}(\boldsymbol{z}(t))=(f_1(z_1(t)),f_2(z_2(t)),\cdots,f_n(z_n(t)))^{\mathrm{T}}$；$\boldsymbol{f}(\boldsymbol{z}(t-\tau))=(f_1(z_1(t-\tau_1)),f_2(z_2(t-\tau_2)),\cdots,f_n(z_n(t-\tau_n)))^{\mathrm{T}}$，$\tau_i=-\log q_i\geqslant 0$，$\tilde{\tau}=\max\limits_{1\leqslant i\leqslant n}\{\tau_i\}=-\log\tilde{q}$；$\boldsymbol{\psi}(s)=\boldsymbol{\varphi}(\mathrm{e}^s)\in C([\log\tilde{q}t_0,\log t_0],\mathbb{R}^n)$。

由假设(H)可知，式 (1) 和式 (2) 的平衡点必存在。设$\boldsymbol{x}^*=(x_1^*,x_2^*,\cdots,x_n^*)^{\mathrm{T}}$和$\boldsymbol{z}^*=(z_1^*,z_2^*,\cdots,z_n^*)^{\mathrm{T}}$为式 (1) 和式 (2) 的平衡点，由平衡点的定义，可算得$\boldsymbol{x}^*=\boldsymbol{z}^*$。因此，可通过探讨式 (2) 的平衡点$\boldsymbol{z}^*$的稳定性来间接讨论式 (1) 的平衡点$\boldsymbol{x}^*$的稳定性情况。

令 $\boldsymbol{y}(t)=\boldsymbol{z}(t)-\boldsymbol{z}^*$，$y_i(t)=z_i(t)-z_i^*$, $i=1,2,\cdots,n$，则式 (2) 改写为：

$$\begin{cases}\dot{\boldsymbol{y}}(t)=\mathrm{e}^t[-\boldsymbol{C}\boldsymbol{y}(t)+\boldsymbol{A}\boldsymbol{g}(\boldsymbol{y}(t))+\\ \boldsymbol{B}\boldsymbol{g}(\boldsymbol{y}(t-\tau))]\qquad t\geqslant\log t_0,\ t\neq t_k\\ \Delta\boldsymbol{y}(t_k)=\boldsymbol{E}_k(\boldsymbol{y}(t_k^-))\qquad k\in\mathbb{N}^*\\ \boldsymbol{y}(s)=\boldsymbol{\eta}(s)\qquad s\in[\log\tilde{q}t_0,\log t_0]\end{cases}\tag{3}$$

式中，$\boldsymbol{g}(\boldsymbol{y}(t-\tau))=(g_1(y_1(t-\tau_1)),g_2(y_2(t-\tau_2)),\cdots,g_n(y_n(t-\tau_n)))^{\mathrm{T}}$，$g_i(y_i(t))=f_i(y_i(t)+z_i^*)-f_i(z_i^*)$，且$g_i(0)=0$；$\boldsymbol{\eta}(s)=\boldsymbol{\psi}(s)-\boldsymbol{z}^*$。

由 (H) 和$g_i(y_i(t))=f_i(y_i(t)+z_i^*)-f_i(z_i^*)$，可得：

$$0\leqslant\frac{g_i(y_i(t))}{y_i(t)}\leqslant l_i\tag{4}$$

由积分中值定理，可得：

$$0\leqslant\int_0^{y_i(t)}g_i(s)\mathrm{d}s\leqslant g_i(y_i(t))y_i(t)\tag{5}$$

由式 (4) 和式 (5)，可得：

$$l_iy_i(t)\geqslant g_i(y_i(t)),\ \ g_i(y_i(t))y_i(t)\geqslant 0$$

从而得：

$$l_ig_i(y_i(t))y_i(t)\geqslant g_i^2(y_i(t))\qquad\forall y_i(t)\in\mathbb{R},\ i=1,2,\cdots,n\tag{6}$$

同时，由式 (6) 可得：

$$\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DCy}(t)=\sum_{i=1}^{n}g_i(y_i(t))d_ic_iy_i(t)\geqslant \sum_{i=1}^{n}\frac{d_ic_i}{l_i}g_i{}^2(y_i(t))=\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DCL}^{-1}\boldsymbol{g}(\boldsymbol{y}(t)) \tag{7}$$

定义 1　称系统 (1) 的平衡点$\boldsymbol{x}^*$是全局多项式稳定的 (GPS)。若存在$\lambda>0$和$\beta\geqslant 1$，使得：

$$\|\boldsymbol{x}(t)-\boldsymbol{x}^*\|^2\leqslant\beta\|\boldsymbol{\varphi}(s)-\boldsymbol{x}^*\|_{\tilde{q}}^2\left(\frac{t}{t_0}\right)^{-\lambda}\qquad t\geqslant t_0$$

式中，$\|\boldsymbol{\varphi}(s)-\boldsymbol{x}^*\|_{\tilde{q}}^2=\sup\limits_{\tilde{q}t_0\leqslant s\leqslant t_0}\|\boldsymbol{\varphi}(s)-\boldsymbol{x}^*\|^2$。

定义 2　称系统 (2) 的平衡点$\boldsymbol{z}^*$是全局指数稳定的 (GES)。若存在$\lambda>0$和$\beta\geqslant 1$，使得：

$$\|z(t)-\boldsymbol{z}^*\|^2\leqslant\beta\|\boldsymbol{\psi}(s)-\boldsymbol{z}^*\|_{\tilde{\tau}}^2\mathrm{e}^{-\lambda(t-\log t_0)}\qquad t\geqslant t_0$$

式中，$\|\boldsymbol{\psi}(s)-\boldsymbol{z}^*\|_{\tilde{\tau}}^2=\sup\limits_{\log\tilde{q}t_0\leqslant s\leqslant\log t_0}\|\boldsymbol{\psi}(s)-\boldsymbol{z}^*\|^2$。

2　全局多项式稳定性

定理 1　假设 (H) 成立，$x_i(t_k)=\lambda_{ik}x_i(t_k{}^-)$，$\lambda_{ik}^2\leqslant 1$。若存在矩阵$\boldsymbol{P}=(p_{ij})_{n\times n}>0$，$\boldsymbol{M}=\mathrm{diag}(m_1,m_2,\cdots,m_n)>0$，$\boldsymbol{D}=\mathrm{diag}(d_1,d_2,\cdots,d_n)>0$，常数$\sigma$满足$0<(\sigma-1)<2\min\{c_i\}$，使得：

$$\boldsymbol{\varXi}=\begin{pmatrix}(\sigma-1)\boldsymbol{P}-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}+\frac{1}{2}(\sigma-1)\boldsymbol{D} & \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\frac{1}{2}(\sigma-1)\boldsymbol{D} & -\boldsymbol{DCL}^{-1}+\boldsymbol{M}+\boldsymbol{DA} & \frac{1}{2}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{B}\\ \boldsymbol{P}^{\mathrm{T}}\boldsymbol{B} & \frac{1}{2}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{D} & -\tilde{q}^{-\sigma}\boldsymbol{M}\end{pmatrix}<0$$

则系统 (1) 的平衡点$\boldsymbol{x}^*$是 GPS。

证明：考虑如下 Lyapunov 泛函：

$$V(t)=\mathrm{e}^{(\sigma-1)t}\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{Py}(t)+\sum_{i=1}^{n}\mathrm{e}^{(\sigma-1)t}d_i\int_0^{y_i(t)}g_i(s)\mathrm{d}s+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\mathrm{e}^{\sigma\theta}m_ig_i^2(y_i(\theta))\mathrm{d}\theta \tag{8}$$

式中，$\boldsymbol{P}=(p_{ij})_{n\times n}>0,\ \sigma>1,\ d_i>0,\ m_i>0,\ i=1,2,\cdots,n$。

当$t\neq t_k$时，将式 (8) 沿系统 (3) 对t进行求导，得：

$$\begin{aligned}&\dot{V}(t)=\mathrm{e}^{(\sigma-1)t}[(\sigma-1)\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{Py}(t)+2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{y}}(t)+\\&\sum_{i=1}^{n}(\sigma-1)d_i\int_0^{y_i(t)}g_i(s)\mathrm{d}s]+\mathrm{e}^{(\sigma-1)t}\sum_{i=1}^{n}g_i(y_i(t))d_i\dot{y}_i(t)+\\&\sum_{i=1}^{n}[\mathrm{e}^{\sigma t}m_ig_i{}^2(y_i(t))-\mathrm{e}^{\sigma(t-\tau_i)}m_ig_i{}^2(y_i(t-\tau_i))]\leqslant\\&\mathrm{e}^{(\sigma-1)t}[(\sigma-1)\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{Py}(t)+2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{y}}(t)+\\&\sum_{i=1}^{n}(\sigma-1)g_i(y_i(t))d_iy_i(t)]+\\&\mathrm{e}^{(\sigma-1)t}\sum_{i=1}^{n}g_i(y_i(t))d_i\dot{y}_i(t)+\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{Mg}(\boldsymbol{y}(t))-\\&\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t-\tau))\mathrm{e}^{-\sigma\tilde{\tau}}\boldsymbol{Mg}(\boldsymbol{y}(t-\tau))=\\&\mathrm{e}^{(\sigma-1)t}\boldsymbol{y}^{\mathrm{T}}(t)(\sigma-1)\boldsymbol{Py}(t)+2\mathrm{e}^{\sigma t}\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}(-\boldsymbol{Cy}(t)+\boldsymbol{Ag}(\boldsymbol{y}(t))+\\&\boldsymbol{Bg}(\boldsymbol{y}(t-\tau)))\ +(\sigma-1)\mathrm{e}^{(\sigma-1)t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{Dy}(t)+\\&\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{D}(-\boldsymbol{Cy}(t)+\boldsymbol{Ag}(\boldsymbol{y}(t))+\boldsymbol{Bg}(\boldsymbol{y}(t-\tau)))+\\&\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{Mg}(\boldsymbol{y}(t))-\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t-\tau))\mathrm{e}^{-\sigma\tilde{\tau}}\boldsymbol{Mg}(\boldsymbol{y}(t-\tau))\leqslant\\&\mathrm{e}^{\sigma t}\Big\{\boldsymbol{y}^{\mathrm{T}}(t)[(\sigma-1)\boldsymbol{P}-\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C}-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}]\boldsymbol{y}(t)+\\&2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{PAg}(\boldsymbol{y}(t))+2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{PBg}(\boldsymbol{y}(t-\tau))+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))(\sigma-1)\boldsymbol{Dy}(t)-\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DCy}(t)+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DAg}^{\mathrm{T}}(\boldsymbol{y}(t))+\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DBg}(\boldsymbol{y}(t-\tau))+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{Mg}(\boldsymbol{y}(t))-\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t-\tau))\mathrm{e}^{-\sigma\tilde{\tau}}\boldsymbol{Mg}(\boldsymbol{y}(t-\tau))\Big\}\end{aligned} \tag{9}$$

由式 (9) 和式 (7)，得：

$$\begin{aligned}&\dot{V}(t)\leqslant\mathrm{e}^{\sigma t}\Big\{\boldsymbol{y}^{\mathrm{T}}(t)[(\sigma-1)\boldsymbol{P}-\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C}-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}]\boldsymbol{y}(t)+\\&2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{PAg}(\boldsymbol{y}(t))+2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{PBg}(\boldsymbol{y}(t-\tau))+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))(\sigma-1)\boldsymbol{Dy}(t)-\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DCL}^{-1}\boldsymbol{g}(\boldsymbol{y}(t))+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DAg}(\boldsymbol{y}(t))+\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DBg}(\boldsymbol{y}(t-\tau)))+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{Mg}(\boldsymbol{y}(t))-\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t-\tau))(\mathrm{e}^{-\sigma\tilde{\tau}}\boldsymbol{M})\boldsymbol{g}(\boldsymbol{y}(t-\tau))\Big\}=\\&\mathrm{e}^{\sigma t}\Big\{\boldsymbol{y}^{\mathrm{T}}(t)[(\sigma-1)\boldsymbol{P}-\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C}-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}]\boldsymbol{y}(t)+\\&2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{PAg}(\boldsymbol{y}(t))+2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{PBg}(\boldsymbol{y}(t-\tau))+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))(\sigma-1)\boldsymbol{Dy}(t)-\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DCL}^{-1}\boldsymbol{g}(\boldsymbol{y}(t))+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DAg}(\boldsymbol{y}(t))+\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{DBg}(\boldsymbol{y}(t-\tau)))+\\&\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{Mg}(\boldsymbol{y}(t))-\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t-\tau))(\mathrm{e}^{-\sigma\tilde{\tau}}\boldsymbol{M})\boldsymbol{g}(\boldsymbol{y}(t-\tau))\Big\}=\\&\mathrm{e}^{\sigma t}\boldsymbol{\xi}^{\mathrm{T}}(t)\boldsymbol{\varXi\xi}(t)\end{aligned}$$

式中，$\xi=\left(y^{\mathrm{T}}(t)\ g^{\mathrm{T}}(y(t))\ g^{\mathrm{T}}(y(t-\tau))\right)^{\mathrm{T}}$。由已知$\varXi<0$，可得：

$$\dot{V}(t)\leqslant 0\qquad t\neq t_k \tag{10}$$

另一方面，当$t\neq t_k$时，由式 (8)，得：

$$\begin{aligned}&V(t_k)=\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{e}^{(\sigma-1)t_k}y_i(t_k)p_{ij}y_j(t_k)+\\&\sum_{i=1}^{n}\mathrm{e}^{(\sigma-1)t_k}d_i\int_0^{y_i(t_k)}g_i(s)\mathrm{d}s+\sum_{i=1}^{n}\int_{t_k-\tau_i}^{t_k}\mathrm{e}^{\sigma\theta}m_ig_i^2(y_i(\theta))\mathrm{d}\theta=\\&\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{e}^{(\sigma-1)t_k{}^-}\lambda_{ik}^2y_i(t_k^-)p_{ij}y_j(t_k{}^-)+\\&\sum_{i=1}^{n}\mathrm{e}^{(\sigma-1)t_k{}^-}d_i\int_0^{\lambda_{ik}y_i(t_k{}^-)}g_i(s)\mathrm{d}s+\end{aligned}$$

$$\sum_{i=1}^{n}\int_{t_k^- -\tau_i}^{t_k^-} e^{\sigma\theta} m_i g_i^2(y_i(\theta))d\theta$$

由于$\lambda_{ik}^2 \leqslant 1$，则$\int_0^{\lambda_{ik}y_i(t_k^-)} g_i(s)ds \leqslant \int_0^{y_i(t_k^-)} g_i(s)ds$。于是：

$$V(t_k) \leqslant V(t_k^-) = e^{(\sigma-1)t_k^-}\boldsymbol{y}^T(t_k^-)\boldsymbol{P}\boldsymbol{y}(t_k^-)+ \sum_{i=1}^{n} e^{(\sigma-1)t_k^-} d_i \int_0^{y_i(t_k^-)} g_i(s)ds + \sum_{i=1}^{n}\int_{t_k^- -\tau_i}^{t_k^-} e^{\sigma\theta} m_i g_i^2(y_i(\theta))d\theta \tag{11}$$

由 (10) 和 (11)，当$t \geqslant \log t_0$时，得：

$$V(t) \leqslant V(\log t_0),\ t \in (t_{k-1}, t_k],\ k \in \mathbb{N}^*$$

当且仅当$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{y}(t)) = \boldsymbol{g}(\boldsymbol{y}(t-\tau))$时，$\dot{V}(t) = 0$。且：

$$V(\log t_0) = e^{(\sigma-1)\log t_0}\boldsymbol{y}^T(\log t_0)\boldsymbol{P}\boldsymbol{y}(\log t_0)+ \sum_{i=1}^{n} e^{(\sigma-1)\log t_0} d_i \int_0^{y_i(\log t_0)} g_i(s)ds+ \sum_{i=1}^{n}\int_{\log q_i t_0}^{\log t_0} e^{\sigma\theta} m_i g_i^2(y_i(\theta))d\theta \leqslant$$

$$e^{(\sigma-1)\log t_0}y^T(\log t_0)\boldsymbol{P}\boldsymbol{y}(\log t_0)+ e^{(\sigma-1)\log t_0}\sum_{i=1}^{n} d_i g_i(y_i(\log t_0))y_i(\log t_0)+ \sum_{i=1}^{n}\int_{\log q_i t_0}^{\log t_0} e^{\sigma\theta} m_i l_i^2 y_i^2(\theta)d\theta \leqslant$$

$$e^{(\sigma-1)\log t_0}\Big(\boldsymbol{y}^T(\log t_0)\boldsymbol{P}\boldsymbol{y}(\log t_0) + \boldsymbol{y}^T(\log t_0)\boldsymbol{LD}\boldsymbol{y}(\log t_0)\Big)+ \max_{1\leqslant i\leqslant n}\left\{m_i l_i^2 t_0^{\sigma}\sigma^{-1}\right\}\sum_{i=1}^{n}\sup_{\log q_i t_0\leqslant\theta\leqslant\log t_0} y_i^2(\theta) \leqslant$$

$$e^{(\sigma-1)\log t_0}\lambda_M(\boldsymbol{P}+\boldsymbol{LD})\|\boldsymbol{y}(\log t_0)\|^2+ \max_{1\leqslant i\leqslant n}\left\{m_i l_i^2 t_0^{\sigma}\sigma^{-1}\right\}\sum_{i=1}^{n}\sup_{\log q_i t_0\leqslant\theta\leqslant\log t_0} y_i^2(\theta) \leqslant$$

$$e^{(\sigma-1)\log t_0}\lambda_M(\boldsymbol{P}+\boldsymbol{LD})\|\boldsymbol{y}(\log t_0)\|^2+ \max_{1\leqslant i\leqslant n}\left\{m_i l_i^2 t_0^{\sigma}\sigma^{-1}\right\}\sup_{\log \tilde{q} t_0\leqslant\theta\leqslant\log t_0}\sum_{i=1}^{n} y_i^2(\theta) \leqslant$$

$$e^{(\sigma-1)\log t_0}\Big\{\lambda_M(\boldsymbol{P}+\boldsymbol{LD})+ \max_{1\leqslant i\leqslant n}[m_i l_i^2 t_0^{\sigma}\sigma^{-1}]\Big\}\sup_{\log \tilde{q} t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{y}(\theta)\|^2 \leqslant$$

$$e^{(\sigma-1)\log t_0}\Big\{\lambda_M(\boldsymbol{P}+\boldsymbol{LD})+ \max_{1\leqslant i\leqslant n}[m_i l_i^2 t_0^{\sigma}\sigma^{-1}]\Big\}\sup_{\log \tilde{q} t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{\eta}(\theta)\|^2$$

又因为：

$$e^{(\sigma-1)t}\boldsymbol{y}^T(t)\boldsymbol{P}\boldsymbol{y}(t) \leqslant V(t) \leqslant V(\log t_0)$$

于是，得：

$$\lambda_m(\boldsymbol{P})e^{(\sigma-1)t}\|\boldsymbol{y}(t)\|^2 \leqslant e^{(\sigma-1)t}\boldsymbol{y}^T(t)\boldsymbol{P}\boldsymbol{y}(t) \leqslant e^{(\sigma-1)\log t_0}\Big\{\lambda_M(\boldsymbol{P}+\boldsymbol{LD})+ \max_{1\leqslant i\leqslant n}[m_i l_i^2 t_0^{\sigma}\sigma^{-1}]\Big\}\sup_{\log \tilde{q} t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{\eta}(\theta)\|^2$$

所以：

$$\|\boldsymbol{y}(t)\|^2 \leqslant \beta \sup_{\log \tilde{q} t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{\eta}(\theta)\|^2 e^{-\lambda(t-\log t_0)} \tag{12}$$

式中，$\lambda = \sigma - 1 > 0$；$\beta = \dfrac{1}{\lambda_m(\boldsymbol{P})}\{\lambda_M(\boldsymbol{P}+\boldsymbol{LD})+ \max\limits_{1\leqslant i\leqslant n}\{m_i l_i^2 t_0^{\sigma}\sigma^{-1}\} \geqslant 1$。

从而，得：

$$\|z(t) - z^*\|^2 \leqslant \beta \sup_{\log \tilde{q} t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{\psi}(\theta) - z^*\|^2 e^{-\lambda(t-\log t_0)} \tag{13}$$

再将$\boldsymbol{x}^* = z^*$和$z(t) = \boldsymbol{x}(e^t)$代入式 (13)，得：

$$\|\boldsymbol{x}(e^t) - \boldsymbol{x}^*\|^2 \leqslant \beta \sup_{\log \tilde{q} t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{\varphi}(e^{\theta}) - \boldsymbol{x}^*\|^2 e^{-\lambda(t-\log t_0)} \tag{14}$$

在式 (14) 中，令$e^t = \gamma$，则$t = \log\gamma \geqslant 0,\ t \geqslant t_0$。再令$e^{\theta} = \xi$，则$\theta \in [\log \tilde{q}t_0, \log t_0]$，$\xi \in [qt_0,\ t_0]$，有：

$$\|\boldsymbol{x}(\gamma) - \boldsymbol{x}^*\|^2 \leqslant \beta \sup_{\tilde{q}t_0\leqslant\xi\leqslant t_0}\|\boldsymbol{\varphi}(\xi) - \boldsymbol{x}^*\|^2 e^{-\lambda(\log\gamma-\log t_0)}$$

再取$\gamma = t$，有：

$$\|\boldsymbol{x}(t) - \boldsymbol{x}^*\|^2 \leqslant \beta \sup_{\tilde{q}t_0\leqslant\xi\leqslant t_0}\|\boldsymbol{\varphi}(\xi) - \boldsymbol{x}^*\|^2 \left(\frac{t}{t_0}\right)^{-\lambda} \quad t \geqslant t_0$$

根据定义 1，可知系统 (1) 的平衡点x^*是 GPS。

由式 (13) 和定义 2，可得下面结果。

定理 2 假设 (H) 成立，$x_i(t_k) = \lambda_{ik}x_i(t_k^-)$，$\lambda_{ik}^2 \leqslant 1$。若存在矩阵$\boldsymbol{P} = (p_{ij})_{n\times n} > 0$，$\boldsymbol{M} = \mathrm{diag}(m_1, m_2, \cdots, m_n) > 0$，$\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \cdots, d_n) > 0$，和常数$\sigma$满足$0 < (\sigma - 1) < 2\min\{c_i\}$，使得：

$$\boldsymbol{\varXi} = \begin{pmatrix} (\sigma-1)\boldsymbol{P}-\boldsymbol{C}^T\boldsymbol{P}-\boldsymbol{P}^T\boldsymbol{C} & \boldsymbol{P}^T\boldsymbol{A} + \frac{1}{2}(\sigma-1)\boldsymbol{D} & \boldsymbol{B}^T\boldsymbol{P} \\ \boldsymbol{A}^T\boldsymbol{P} + \frac{1}{2}(\sigma-1)\boldsymbol{D} & -\boldsymbol{DCL}^{-1}+\boldsymbol{M}+\boldsymbol{DA} & \frac{1}{2}\boldsymbol{D}^T\boldsymbol{B} \\ \boldsymbol{P}^T\boldsymbol{B} & \frac{1}{2}\boldsymbol{B}^T\boldsymbol{D} & -e^{\sigma\bar{\tau}}\boldsymbol{M} \end{pmatrix} < 0$$

则系统 (2) 的平衡点z^*是 GES。

定理 3 假设 (H) 成立，$x_i(t_k) = \lambda_{ik}x_i(t_k^-)$，$\lambda_{ik}^2 \leqslant 1$。当$q_i = q,\ i = 1, 2, \cdots, n$时，若存在矩阵$\boldsymbol{P} = (p_{ij})_{n\times n} > 0$，$\boldsymbol{M} > 0$，$\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \cdots, d_n) > 0$，和常数$\sigma$满足$0 < (\sigma - 1) < 2\min\{c_i\}$使得：

$$\boldsymbol{\Xi}=\begin{pmatrix}(\sigma-1)\boldsymbol{P}-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}+\frac{1}{2}(\sigma-1)\boldsymbol{D} & \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\frac{1}{2}(\sigma-1)\boldsymbol{D} & -\boldsymbol{DCL}^{-1}+\boldsymbol{M}+\boldsymbol{DA} & \frac{1}{2}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{B}\\ \boldsymbol{P}^{\mathrm{T}}\boldsymbol{B} & \frac{1}{2}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{D} & -q^{-\sigma}\boldsymbol{M}\end{pmatrix}<0$$

则系统 (1) 的平衡点$\boldsymbol{x}^*$是 GPS。

证明：当$q_i=q,\ i=1,2,\cdots,n$时，则有系统 (2) 和 (3) 中的$\tau_i=\tau,\ i=1,2,\cdots,n$。考虑如下 Lyapunov 泛函：

$$V(t)=\mathrm{e}^{(\sigma-1)t}\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{y}(t)+\sum_{i=1}^{n}\mathrm{e}^{(\sigma-1)t}d_i\int_0^{y_i(t)}g_i(s)\mathrm{d}s+\int_{t-\tau}^{t}\mathrm{e}^{\sigma\theta}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(\theta))\boldsymbol{M}\boldsymbol{g}(\boldsymbol{y}(\theta))\mathrm{d}\theta \tag{15}$$

式中，$\boldsymbol{P}=(p_{ij})_{n\times n}>0,\ \boldsymbol{M}>0,\ \sigma>1,\ d_i>0,\ i=1,2,\cdots,n$。

当$t\neq t_k$时，将式 (14) 沿系统 (3) 对t进行求导，得：

$$\begin{aligned}\dot{V}(t)=&\mathrm{e}^{(\sigma-1)t}[(\sigma-1)\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{y}(t)+2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{y}}(t)+\sum_{i=1}^{n}(\sigma-1)d_i\int_0^{y_i(t)}g_i(s)\mathrm{d}s]+\\&\mathrm{e}^{(\sigma-1)t}\sum_{i=1}^{n}g_i(y_i(t))d_i\dot{y}_i(t)+\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{M}\boldsymbol{g}(\boldsymbol{y}(t))-\\&\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t-\tau))\mathrm{e}^{-\sigma\tau}\boldsymbol{M}\boldsymbol{g}(\boldsymbol{y}(t-\tau))\leqslant\\&\mathrm{e}^{(\sigma-1)t}[(\sigma-1)\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{y}(t)+2\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{y}}(t)+\sum_{i=1}^{n}(\sigma-1)g_i(y_i(t))d_iy_i(t)]+\\&\mathrm{e}^{(\sigma-1)t}\sum_{i=1}^{n}g_i(y_i(t))d_i\dot{y}_i(t)+\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{M}\boldsymbol{g}(\boldsymbol{y}(t))-\\&\mathrm{e}^{\sigma t}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t-\tau))\mathrm{e}^{-\sigma\tau}\boldsymbol{M}\boldsymbol{g}(\boldsymbol{y}(t-\tau))\leqslant\mathrm{e}^{\sigma t}\boldsymbol{\xi}^{\mathrm{T}}(t)\boldsymbol{\Xi}\boldsymbol{\xi}(t)\end{aligned}$$

中间过程与定理 1 的证明类似，这里省略。其中$\boldsymbol{\xi}(t)=(\boldsymbol{y}^{\mathrm{T}}(t),\boldsymbol{g}^{\mathrm{T}}(y(t)),\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(t-\tau)))$。由定理 3 的条件$\boldsymbol{\Xi}<0$，可得：

$$\dot{V}(t)\leqslant 0\quad t\neq t_k \tag{16}$$

另一方面，当$t\neq t_k$时，由式 (15)，得：

$$\begin{aligned}V(t_k)=&\mathrm{e}^{(\sigma-1)t_k}(\sum_{i=1}^{n}\sum_{j=1}^{n}y_i(t_k)P_{ij}y_j(t_k)+\sum_{i=1}^{n}d_i\int_0^{y_i(t_k)}g_i(s)\mathrm{d}s)+\int_{t_k-\tau}^{t_k}\mathrm{e}^{\sigma\theta}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(\theta))\boldsymbol{M}\boldsymbol{g}(\boldsymbol{y}(\theta))\mathrm{d}\theta=\\&\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{e}^{(\sigma-1)t_k^-}\lambda_{ik}^2y_i(t_k^-)P_{ij}y_j(t_k^-)+\sum_{i=1}^{n}\mathrm{e}^{(\sigma-1)t_k^-}d_i\int_0^{\lambda_{ik}y_i(t_k^-)}g_i(s)\mathrm{d}s+\\&\int_{t_k^--\tau}^{t_k^-}\mathrm{e}^{\sigma\theta}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(\theta))\boldsymbol{M}\boldsymbol{g}(y(\theta))\mathrm{d}\theta\end{aligned}$$

由于$\lambda_{ik}^2\leqslant 1$，则$\int_0^{\lambda_{ik}y_i(t_k^-)}g_i(s)\mathrm{d}s\leqslant\int_0^{y_i(t_k^-)}g_i(s)\mathrm{d}s$。于是，得：

$$\begin{aligned}V(t_k)\leqslant&V(t_k^-)=\\&\mathrm{e}^{(\sigma-1)t_k^-}\boldsymbol{y}^{\mathrm{T}}(t_k^-)\boldsymbol{P}\boldsymbol{y}(t_k^-)+\sum_{i=1}^{n}\mathrm{e}^{(\sigma-1)t_k^-}d_i\int_0^{y_i(t_k^-)}g_i(s)\mathrm{d}s+\\&\int_{t_k^--\tau}^{t_k^-}\mathrm{e}^{\sigma\theta}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(\theta))\boldsymbol{M}\boldsymbol{g}(\boldsymbol{y}(\theta))\mathrm{d}\theta\end{aligned} \tag{17}$$

由式 (16) 和式 (17) 可知，当$t\geqslant\log t_0$时，得：

$$V(t)\leqslant V(\log t_0)\quad t\in(t_{k-1},t_k],\ k\in\mathbb{N}^*$$

当且仅当$\boldsymbol{y}(t)=\boldsymbol{g}(\boldsymbol{y}(t))=\boldsymbol{g}(\boldsymbol{y}(t-\tau))$时，$\dot{V}(t)=0$。且：

$$\begin{aligned}V(\log t_0)=&\mathrm{e}^{(\sigma-1)\log t_0}\boldsymbol{y}^{\mathrm{T}}(\log t_0)\boldsymbol{P}\boldsymbol{y}(\log t_0)+\sum_{i=1}^{n}\mathrm{e}^{(\sigma-1)\log t_0}d_i\int_0^{y_i(\log t_0)}g_i(s)\mathrm{d}s+\\&\int_{\log\tilde{q}t_0}^{\log t_0}\mathrm{e}^{\sigma\theta}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y}(\theta))\boldsymbol{M}\boldsymbol{g}(\boldsymbol{y}(\theta))\mathrm{d}\theta\leqslant\\&\mathrm{e}^{(\sigma-1)\log t_0}\boldsymbol{y}^{\mathrm{T}}(\log t_0)\boldsymbol{P}\boldsymbol{y}(\log t_0)+\sum_{i=1}^{n}\mathrm{e}^{(\sigma-1)\log t_0}d_ig_i(y_i(\log t_0))y_i(\log t_0)+\\&\int_{\log\tilde{q}t_0}^{\log t_0}\mathrm{e}^{\sigma\theta}\boldsymbol{y}^{\mathrm{T}}(\theta)\boldsymbol{LML}\boldsymbol{y}(\theta)\mathrm{d}\theta\leqslant\\&\mathrm{e}^{(\sigma-1)\log t_0}\left(\boldsymbol{y}^{\mathrm{T}}(\log t_0)\boldsymbol{P}\boldsymbol{y}(\log t_0)+\boldsymbol{y}^{\mathrm{T}}(\log t_0)\boldsymbol{LD}\boldsymbol{y}(\log t_0)\right)+\\&\left[\lambda_M(\boldsymbol{LML})t_0^{\sigma}\sigma^{-1}\right]\sup_{\log\tilde{q}t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{y}(\theta)\|^2\leqslant\\&\mathrm{e}^{(\sigma-1)\log t_0}\lambda_M(\boldsymbol{P}+\boldsymbol{LD})\|\boldsymbol{y}(\log t_0)\|^2+\left[\lambda_M(\boldsymbol{LML})t_0^{\sigma}\sigma^{-1}\right]\sup_{\log\tilde{q}t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{y}(\theta)\|^2\leqslant\\&\mathrm{e}^{(\sigma-1)\log t_0}\{\lambda_M(\boldsymbol{P}+\boldsymbol{LD})+\lambda_M(\boldsymbol{LML})t_0^{\sigma}\sigma^{-1}\}\sup_{\log\tilde{q}t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{\eta}(\theta)\|^2\end{aligned}$$

又因为：

$$\mathrm{e}^{(\sigma-1)t}\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{y}(t)\leqslant V(t)\leqslant V(\log t_0)$$

于是，有：

$$\begin{aligned}\lambda_m(\boldsymbol{P})\mathrm{e}^{(\sigma-1)t}\|\boldsymbol{y}(t)\|^2\leqslant&\mathrm{e}^{(\sigma-1)t}\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{y}(t)\leqslant\\&\mathrm{e}^{(\sigma-1)\log t_0}\{\lambda_M(\boldsymbol{P}+\boldsymbol{LD})+\lambda_M(\boldsymbol{LQL})t_0^{\sigma}\sigma^{-1}\}\sup_{\log\tilde{q}t_0\leqslant\theta\leqslant\log t_0}\|\boldsymbol{\eta}(\theta)\|^2\end{aligned}$$

所以：

$$\|\boldsymbol{y}(t)\|^2 \leqslant \frac{1}{\lambda_m(\boldsymbol{P})}\{\lambda_M(\boldsymbol{P}+\boldsymbol{LD})+$$

$$\lambda_M(\boldsymbol{LQL})t_0^{\sigma}\sigma^{-1}\} \sup_{\log\tilde{q}t_0\leqslant\theta\leqslant\log t_0} \|\boldsymbol{\eta}(\theta)\|^2 \mathrm{e}^{-\lambda(t-\log t_0)} \quad (18)$$

其中$\lambda=\sigma-1>0$。取：

$$\beta=\frac{1}{\lambda_m(\boldsymbol{P})}\{\lambda_M(\boldsymbol{P}+\boldsymbol{LD})+\lambda_M(\boldsymbol{LQL})(1-\mathrm{e}^{-\sigma\tau})\sigma^{-1}\}\geqslant 1$$

因此，由 (18)，得：

$$\|\boldsymbol{y}(t)\|^2 \leqslant \beta \sup_{\log\tilde{q}t_0\leqslant\theta\leqslant\log t_0} \|\boldsymbol{\eta}(\theta)\|^2 \mathrm{e}^{-\lambda(t-\log t_0)}$$

余下部分与定理 1 相同。

在定理 1、2 和 3 中，当$\sigma=1$时，则全局多项式稳定性和指数稳定性都退化为全局渐近稳定性。

推论 1 假设(H)成立，$x_i(t_k)=\lambda_{ik}x_i(t_k^-)$，$\lambda_{ik}^2\leqslant 1$。若存在矩阵$\boldsymbol{P}>0$，$\boldsymbol{M}=\mathrm{diag}(m_1,m_2,\cdots,m_n)>0$，$\boldsymbol{D}=\mathrm{diag}(d_1,d_2,\cdots,d_n)>0$，使得：

$$\boldsymbol{\varXi}=\begin{pmatrix} -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{P}^{\mathrm{T}}\boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} \\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} & -\boldsymbol{DCL}^{-1}+\boldsymbol{M}+\boldsymbol{DA} & \frac{1}{2}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{B} \\ \boldsymbol{P}^{\mathrm{T}}\boldsymbol{B} & \frac{1}{2}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{D} & -\tilde{q}^{-1}\boldsymbol{M} \end{pmatrix}<0$$

则系统 (1) 和 (2) 的平衡点$\boldsymbol{x}^*$和$\boldsymbol{z}^*$是全局渐近稳定的。

推论 2 假设 (H) 成立，$x_i(t_k)=\lambda_{ik}x_i(t_k^-)$，$\lambda_{ik}^2\leqslant 1$。当$q_i=q, i=1,2,\cdots,n$时，若存在矩阵$\boldsymbol{P}>0$，$\boldsymbol{M}>0$和$\boldsymbol{D}=\mathrm{diag}(d_1,d_2,\cdots,d_n)>0$，使得：

$$\boldsymbol{\varXi}=\begin{pmatrix} -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{P}^{\mathrm{T}}\boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} \\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} & -\boldsymbol{DCL}^{-1}+\boldsymbol{M}+\boldsymbol{DA} & \frac{1}{2}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{B} \\ \boldsymbol{P}^{\mathrm{T}}\boldsymbol{B} & \frac{1}{2}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{D} & -q^{-1}\boldsymbol{M} \end{pmatrix}<0$$

则系统 (1) 和 (2) 的平衡点$\boldsymbol{x}^*$和$\boldsymbol{z}^*$都是全局渐近稳定的 (GAS)。

虽然系统 (1) 和 (2) 等价，且具有相同的平衡点$\boldsymbol{x}^*=\boldsymbol{z}^*$，但是他们的稳定性却不同，系统 (2) 的平衡点$\boldsymbol{z}^*$是 GES，系统 (1) 的平衡点$\boldsymbol{x}^*$却是 GPS。

当$\lambda=0.5$、1、1.5、3时，函数$y=t^{-\lambda}$和$y=\mathrm{e}^{-\lambda t}$的函数图像，如图 1 所示，可以直观地看到多项式稳定性的收敛速度比指数稳定性的慢一些。

当系统 (1) 无脉冲影响时，本文所得结论仍然成立。

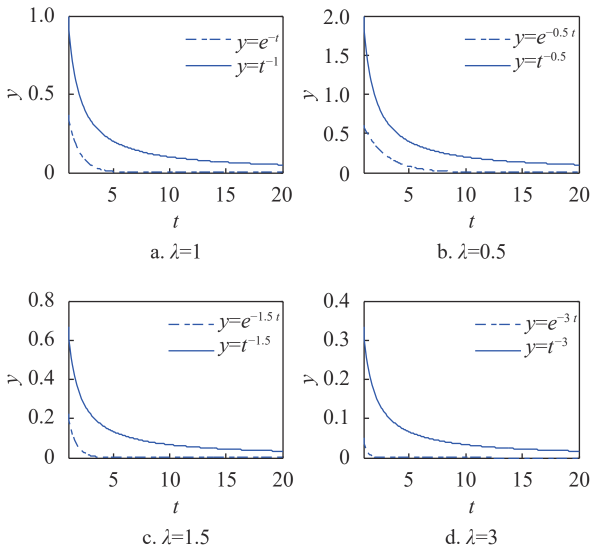

图 1 函数$y=t^{-\lambda}$和$y=\mathrm{e}^{-\lambda t}$的图像

3 数值算例

例 1 考虑二维如下 RNNs：

$$\begin{cases} \dot{x}_1(t)=-4.6x_1(t)-2f_1(x_1(t))+f_2(x_2(t))+ \\ \quad 1.5f_1(x_1(0.4t))+0.2f_2(x_2(0.8t))+u_1 \quad t\neq t_k \\ \dot{x}_2(t)=-3x_2(t)+f_1(x_1(t))-3f_2(x_2(t))+ \\ \quad 1.4f_1(x_1(0,4t))+2.3f_2(x_2(0.8t))+u_2 \quad t\neq t_k \\ x_1(t_k)=E_{1k}x_1(t_k^-) \quad k\in\mathbb{N}^* \\ x_2(t_k)=E_{2k}x_2(t_k^-) \quad k\in\mathbb{N}^* \end{cases} \quad (19)$$

式中，$f_1(x_1(t))=0.25(|x_1(t)+1|-|x_1(t)-1|)$；$f_2(x_2(t))=\sin(0.25x_2(t))+0.25x_2(t)$；$q_1=0.4$；$q_2=0.8$；$\boldsymbol{u}=(0,0)^{\mathrm{T}}$；$l_i=0.5,\ i=1,2$；$\boldsymbol{L}=\mathrm{diag}(0.5,0.5)$；$E_{1k}=-0.23$；$E_{2k}=0.38$。由式 (19)，可知：

$$\boldsymbol{C}=\begin{pmatrix} 4.6 & 0 \\ 0 & 3 \end{pmatrix},\boldsymbol{A}=\begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix},\boldsymbol{B}=\begin{pmatrix} 1.5 & 0.2 \\ 1.4 & 2.3 \end{pmatrix}$$

取$\boldsymbol{P}=\boldsymbol{Q}=\boldsymbol{D}=\mathrm{diag}(1,1)$，$\sigma=1.1$，通过应用 Matlab 计算，得：

$$\boldsymbol{\varXi}=\begin{pmatrix} -9.1000 & 0 & -1.9500 & 1.0000 & 1.5000 & 1.4000 \\ 0 & -5.9000 & 1.0000 & -2.9500 & 0.2000 & 2.3000 \\ -1.9500 & 1.0000 & -10.2000 & 1.0000 & 0.7500 & 0.1000 \\ 1.0000 & -2.9500 & 1.0000 & -8.0000 & 0.7000 & 1.1500 \\ 1.5000 & 1.4000 & 0.7500 & 0.7000 & -2.7399 & 0 \\ 0.2000 & 2.3000 & 0.1000 & 1.1500 & 0 & -2.7399 \end{pmatrix}$$

且λ_{Ξ}= −12.756 7, −10.255 2, −7.726 6, −4.182 5, −2.375 5, −1.383 2，即$\Xi<0$。由定理 1 可知，系统 (19) 是 GPS。系统 (19) 的平衡点为$\boldsymbol{x}^*=(0,0)^{\mathrm{T}}$。时间响应轨线如图 2～图 5 所示。

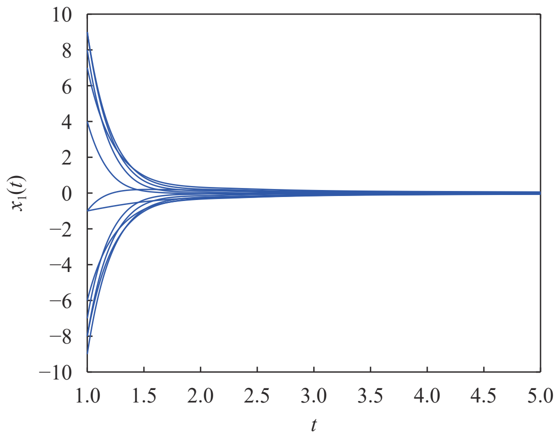

图 2　系统 (19) 无脉冲时，$x_1(t)$的时间响应轨线

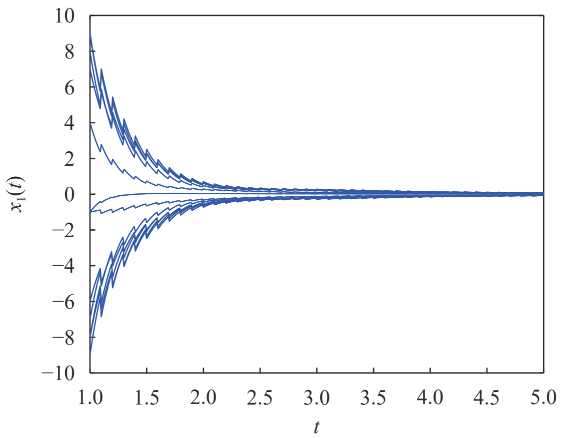

图 3　系统 (19) 带脉冲时，$x_1(t)$的时间响应轨线

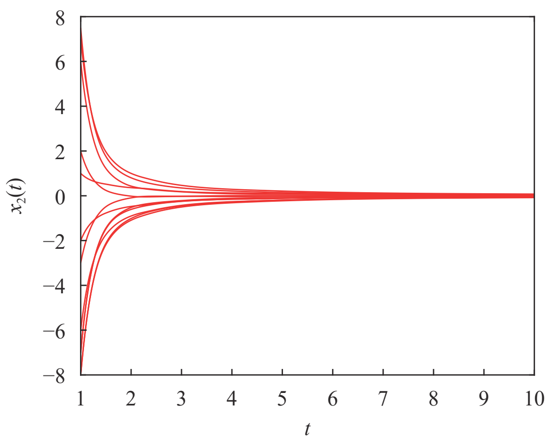

图 4　系统 (19) 无脉冲时，$x_2(t)$的时间响应轨线

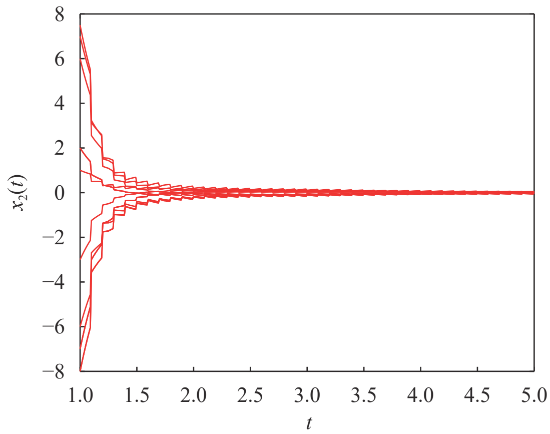

图 5　系统 (19) 带脉冲时，$x_2(t)$的时间响应轨线

例 2　考虑三维具比例时滞 RNNs：

$$\left\{\begin{aligned}&\dot{x}_1(t)=-3.5x_1(t)-2f_1(x_1(t))+1.2f_2(x_2(t))+\\&\quad f_3(x_3(t))+0.5f_1(x_1(0.3t))+0.3f_2(x_2(0.5t)+u_1\\&\dot{x}_2(t)=-5.6x_2(t)+2f_1(x_1(t))-f_2(x_2(t)))+\\&\quad 3.2f_1(x_1(0.3t)+2.4f_2(x_2(0.5t)+u_2\\&\dot{x}_3(t)=-3.7x_3(t)+3f_1(x_1(t))+2f_3(x_3(t))+\\&\quad 3f_2(x_2(0.5t))+1.5f_3(x_3(0.8t))+u_3\\&x_1(t_k)=E_{1k}x_1(t_k^-)\quad k\in\mathbb{N}^*\\&x_2(t_k)=E_{2k}x_2(t_k^-)\quad k\in\mathbb{N}^*\\&x_3(t_k)=E_{3k}x_3(t_k^-)\quad k\in\mathbb{N}^*\end{aligned}\right.\tag{20}$$

式中，$f_i(x_i(t))=\tanh(0.5x_i(t)),\ i=1,2$；$f_3(x_3(t))=\cos(x_3(0.2t))$，$q_1=0.3,q_2=0.5,q_3=0.8$，$\tilde{q}=0.3$，$\boldsymbol{u}=(0,-3,4)^{\mathrm{T}}$，$E_{1k}=0.63$，$E_{2k}=-0.28$，$E_{3k}=0.18$，$k\in\mathbb{N}^*$。Lipschitz 常数为$l_i=0.5,\ i=1,2$，$l_3=0.2$，$\boldsymbol{L}=\mathrm{diag}(0.5,0.5,0.2)$。

由式 (20)，可知：

$$\boldsymbol{C}=\begin{pmatrix}3.5&0&0\\0&5.6&0\\0&0&3.7\end{pmatrix},\ \boldsymbol{A}=\begin{pmatrix}-2&1.2&1\\2&-1&0\\3&0&2\end{pmatrix},$$

$$\boldsymbol{B}=\begin{pmatrix}1.5&1.3&0\\3.2&2.4&0\\0&3&1.5\end{pmatrix}$$

利用 Matlab 找到$\sigma=1.236\,7$和以下矩阵：

$$\boldsymbol{P}=\begin{pmatrix}1.989\,7&0.002\,3&0.104\,5\\0.002\,3&1.002\,3&0.119\,8\\0.104\,5&0.119\,8&0.998\,7\end{pmatrix}$$

$$\boldsymbol{M}=\begin{pmatrix}1.567\,8&0&0\\0&2.089\,7&0\\0&0&3.045\,7\end{pmatrix}$$

$$\boldsymbol{D}=\begin{pmatrix}2.012\,3&0&0\\0&1.478\,5&0\\0&0&3.027\,83\end{pmatrix}$$

得到矩阵$\boldsymbol{\Xi}$，且$\lambda_{\Xi}= -48.370\,1, -21.586\,4, -2.085\,5, -5.302\,1, -17.523\,0, -8.011\,9, -10.002\,8, -14.865\,7, -12.968\,0$, 由此可知$\Xi < 0$，由定理 1，可知系统 (20) 是 GPS。由 Matlab 计算，得系统 (20) 的平衡点为 $\boldsymbol{x}^* = (0.154\,5,\ -0.539\,5, 1.880\,5)^{\mathrm{T}}$。时间响应曲线如图 6～11 所示。

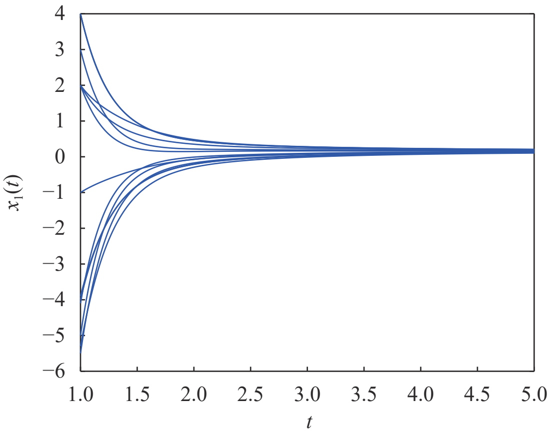

图 6 系统 (20) 无脉冲时，$x_1(t)$的时间响应轨线

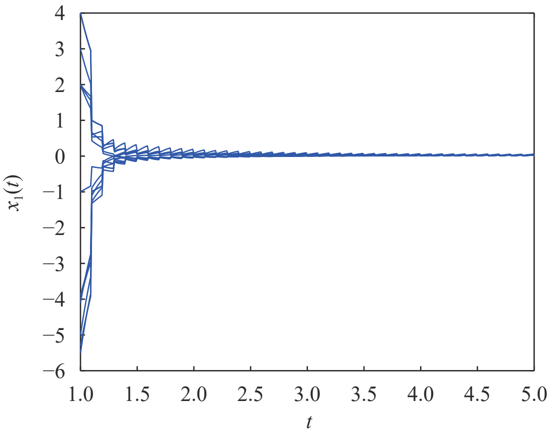

图 7 系统 (20) 带脉冲时，$x_1(t)$的时间响应轨线

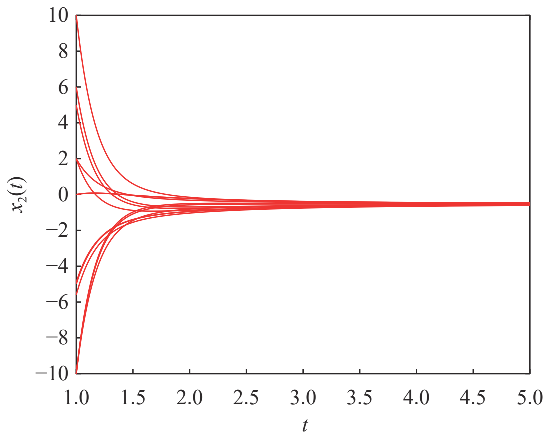

图 8 系统 (20) 无脉冲时，$x_2(t)$的时间响应轨线

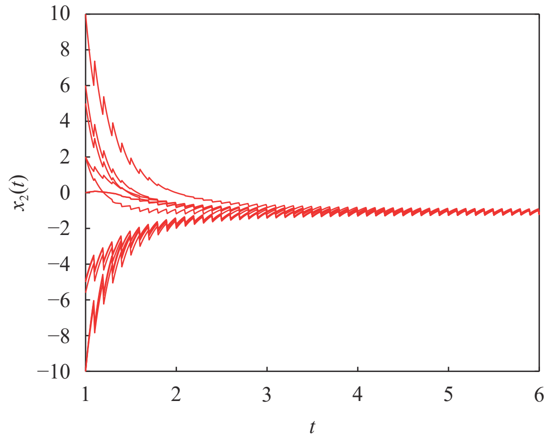

图 9 系统 (20) 带脉冲时，$x_2(t)$的时间响应轨线

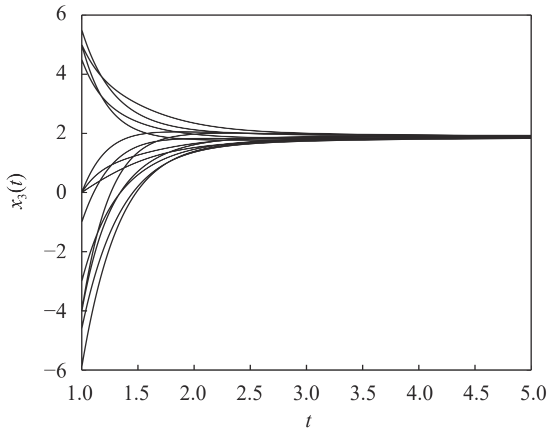

图 10 系统 (20) 无脉冲时，$x_3(t)$的时间响应轨线

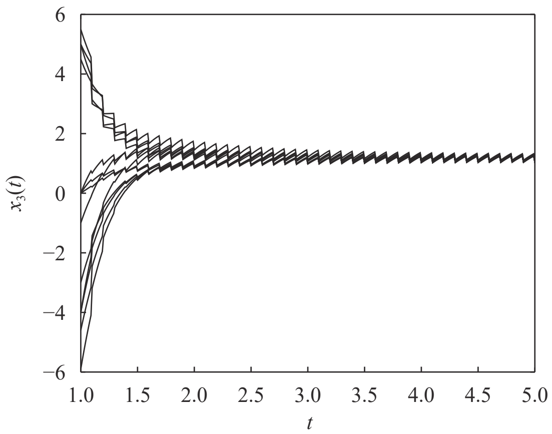

图 11 系统 (20) 带脉冲时，$x_3(t)$的时间响应轨线

4 结束语

本文通过构造合适的 Lyapunov 泛函，研究了一类具比例时滞 IRNNs 的全局多项式稳定性，所得准则是以 LMI 形式给出的，便于应用 Matlab 验证。可调参数的引入使得所得条件的适用范围扩大

了。激活函数较广泛，可以是无界的，也可以是不可微的。本文的研究方法也适合于具比例时滞 RNNs 的多项式周期性、多项式同步性和多项式耗散性等动力学行为的研究。

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