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Algorithms of Decomposition and Reconstruction with Biorthogonal Multiwavelet Packets with Scale=*a**

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Abstract　A method for construction of biorthogonal multiwavelet packets with scale=a is proposed in this paper. They are more flexible in application. The space $L^2(R)$ can be decomposed by using the proposed multiwavelet packets. The algorithms of decomposition and reconstruction of biorthogonal multiwavelet packets are given finally.

Key words　biorthogonality;　multiwavelet;　wavelet packets;　scale;　scaling vector

*a*尺度多重双正交小波包的分解与重构算法

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【摘要】介绍了构造a尺度多重双正交小波包的方法。它在应用上灵活性很强。按此法可构造多种不同的双正交小波包。本文的重点是给出了用此小波包进行分解与重构的算法。

关　键　词　半正交性;　多小波;　尺度;　小波包;　尺度函数向量
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As well known, scalar orthogonal wavelet bases with compactly support have not symmetry. Multiwavelets initiated by Ret.[1] overcome that drawback. Since then multiwavelet have received considerable attention from the wavelet research communities both in theory and in applications such as signal compression and denoising for consideration of multi-pass filter and more flexible wavelets than two-scale wavelets. Ref.[2] introduced the theory of multiwavelet with scale=a and got wonderful results. Ref.[3] composed and studied biorthogonal wavelet.

Wavelet packets were introduced by Refs.[4,5] to improve the poor frequency localization of wavelet bases and thereby provided a more efficient decomposition of signals containing both transient and stationary components. The advantages of wavelet packets and their promising features in application have attracted a great deal of interest and effort in recent years to extensively study them. To only mention a few references here, see Refs.[6-11]. Ref.[5] introduced biorthogonal wavelet packets using splines. Ref.[9] provided a method of construction of orthogonal multiwavelet packets. Ref.[10] discussed the biorthogonal case and studied their properties.

In this paper, we introduce a recipe for construction of biorthogonal multiwavelet packets with scale=a. More biorthogonal multiwavelet packets can be constructed from the same biorthogonal multiwavelet by using the recipe. We can decompose the space $L^2(R)$ by using the multiwavelet packets of this paper. Specially, an important algorithms of decomposition of biorthogonal multiwavelet packets with scale=a is given finally. We adopt symbols same as Ref [11].

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1 Biorthogonal Multiwavelet Packets with Scale=a

Let's begin with some basal knowledge. Then we introduce the construction of biorthogonal multiwavelet packets with scale=a.

A pair of vector functions $\boldsymbol{\Phi}=\{\varphi_1,\varphi_2,\cdots,\varphi_t\}$ and $\tilde{\boldsymbol{\Phi}}=\{\tilde{\varphi}_1,\tilde{\varphi}_2,\cdots,\tilde{\varphi}_t\}$ are said to be biorthogonal scaling vectors if $\langle\boldsymbol{\Phi}(\cdot),\tilde{\boldsymbol{\Phi}}(\cdot-n)\rangle=\delta_{o,n}I_t$ for $n\in Z$. And a pair of vector functions $\boldsymbol{\Psi}=\{\psi_1,\psi_2,\cdots,\psi_{(a-1)t}\}$ and $\tilde{\boldsymbol{\Psi}}=\{\tilde{\psi}_1,\tilde{\psi}_2,\cdots,\tilde{\psi}_{(a-1)t}\}$ are said to be biorthogonal multiwavelets if $\langle\boldsymbol{\Phi}(\cdot),\tilde{\boldsymbol{\Psi}}(\cdot-n)\rangle=\langle\boldsymbol{\Psi}(\cdot),\tilde{\boldsymbol{\Phi}}(\cdot-n)\rangle=O$ and $\langle\boldsymbol{\Psi}(\cdot),\tilde{\boldsymbol{\Psi}}(\cdot-n)\rangle=\delta_{o,n}I_{(a-1)t}$ for $n\in Z$. And there exist some sequences of matrices $\{\boldsymbol{P}_k\},\{\tilde{\boldsymbol{P}}_k\},\{\boldsymbol{Q}_k\}$ and $\{\tilde{\boldsymbol{Q}}_k\}$ such that

$$\boldsymbol{\Phi}=\sum_{k\in Z}\boldsymbol{P}_k\boldsymbol{\Phi}(ax-k),\ \tilde{\boldsymbol{\Phi}}=\sum_{k\in Z}\tilde{\boldsymbol{P}}_k\tilde{\boldsymbol{\Phi}}(ax-k),\boldsymbol{\Psi}=\sum_{k\in Z}\boldsymbol{Q}_k\boldsymbol{\Psi}(ax-k),\ \tilde{\boldsymbol{\Psi}}=\sum_{k\in Z}\tilde{\boldsymbol{Q}}_k\tilde{\boldsymbol{\Psi}}(ax-k)$$

and their Fourier transforms yield

$$\hat{\boldsymbol{\Phi}}(\omega)=\boldsymbol{P}(z)\hat{\boldsymbol{\Phi}}\left(\frac{\omega}{a}\right),\ \hat{\tilde{\boldsymbol{\Phi}}}(\omega)=\tilde{\boldsymbol{P}}(z)\hat{\tilde{\boldsymbol{\Phi}}}\left(\frac{\omega}{a}\right),\ \hat{\boldsymbol{\Psi}}(\omega)=\boldsymbol{Q}(z)\hat{\boldsymbol{\Psi}}\left(\frac{\omega}{a}\right),\ \hat{\tilde{\boldsymbol{\Psi}}}(\omega)=\tilde{\boldsymbol{Q}}(z)\hat{\tilde{\boldsymbol{\Psi}}}\left(\frac{\omega}{a}\right)$$

where:

$$\boldsymbol{P}(z)=\frac{1}{a}\sum_{k\in Z}\boldsymbol{P}_k z^k,\ \tilde{\boldsymbol{P}}(z)=\frac{1}{a}\sum_{k\in Z}\tilde{\boldsymbol{P}}_k z^k,\ \boldsymbol{Q}(z)=\frac{1}{a}\sum_{k\in Z}\boldsymbol{Q}_k z^k,\ \tilde{\boldsymbol{Q}}(z)=\frac{1}{a}\sum_{k\in Z}\tilde{\boldsymbol{Q}}_k z^k,\ z=e^{i\omega/a}$$

are called matrix symbols of $\{\boldsymbol{P}_k\},\{\tilde{\boldsymbol{P}}_k\},\{\boldsymbol{Q}_k\}$ and $\{\tilde{\boldsymbol{Q}}_k\}$ respectively.We define:

$$\varphi_{l;i,j}=a^{j/2}\varphi_l(a^jx-i),\ \tilde{\varphi}_{l;i,j}=a^{j/2}\tilde{\varphi}_l(a^jx-i)\psi_{l;i,j}=a^{j/2}\psi_l(a^jx-i),\ \tilde{\psi}_{l;i,j}=a^{j/2}\tilde{\psi}_l(a^jx-i)$$

$$V_j=clos_{L^2(R)}\langle\varphi_{l;i,j}:1\leq l\leq t,i\in Z\rangle,\ \tilde{V}_j=clos_{L^2(R)}\langle\tilde{\varphi}_{l;i,j}:1\leq l\leq t,i\in Z\rangle$$

$$W_j=clos_{L^2(R)}\langle\psi_{l;i,j}:1\leq l\leq(a-1)t,i\in Z\rangle$$

$$\tilde{W}_j=clos_{L^2(R)}\langle\tilde{\psi}_{l;i,j}:1\leq l\leq(a-1)t,i\in Z\rangle\qquad j\in Z$$

$\tilde{V}_j,\tilde{W}_j$ are called dual spaces of V_j,W_j,respectively. For construction of biorthogonal multiwavelet packets, we divide $\boldsymbol{\Psi}$ and $\tilde{\boldsymbol{\Psi}}$ into $a-1$ function vectors with dimension t arbitrarily as follow

$$\boldsymbol{\Psi}_i=\{\psi_{i_1},\psi_{i_2},\cdots,\psi_{i_t}\},\ \tilde{\boldsymbol{\Psi}}_i=\{\tilde{\psi}_{i_1},\tilde{\psi}_{i_2},\cdots,\tilde{\psi}_{i_t}\}\qquad i=1,2,\cdots,a-1$$

And we divided $\boldsymbol{Q}_k$ and $\tilde{\boldsymbol{Q}}_k$ into $a-1$ $t\times t$ matrices according to the dividing of $\boldsymbol{\Psi}$ and $\tilde{\boldsymbol{\Psi}}$ as follow $\boldsymbol{Q}_k=(\boldsymbol{Q}_k^{(1)\mathrm{T}},\boldsymbol{Q}_k^{(2)\mathrm{T}},\cdots,\boldsymbol{Q}_k^{(a-1)\mathrm{T}})^{\mathrm{T}},\ \tilde{\boldsymbol{Q}}_k=(\tilde{\boldsymbol{Q}}_k^{(1)\mathrm{T}},\tilde{\boldsymbol{Q}}_k^{(2)\mathrm{T}},\cdots,\tilde{\boldsymbol{Q}}_k^{(a-1)\mathrm{T}})^{\mathrm{T}}$, Suppose

$$\boldsymbol{P}_k^{(0)}=\boldsymbol{P}_k,\ \tilde{\boldsymbol{P}}_k^{(0)}=\tilde{\boldsymbol{P}}_k,\ \boldsymbol{P}_k^{(i)}=\boldsymbol{Q}_k^{(i)},\ \tilde{\boldsymbol{P}}_k^{(i)}=\tilde{\boldsymbol{Q}}_k^{(i)}\qquad i=1,2,\cdots,a-1;\ k\in Z$$

$$\boldsymbol{\Psi}_0(x)=\boldsymbol{\Phi}(x),\ \tilde{\boldsymbol{\Psi}}_0(x)=\tilde{\boldsymbol{\Phi}}(x)$$

Definition the vector collections $\{\boldsymbol{\Psi}_{al+i}:l=0,1,\cdots,i=0,1,\cdots,a-1\}$ and $\{\tilde{\boldsymbol{\Psi}}_{al+i}:\ l=0,1,\cdots,i=0,1,\cdots,a-1\}$ are called biorthogonal multiwavelet packets with scale= a associated with $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ respectively, where

$$\boldsymbol{\Psi}_{al+i}=\sum_{k\in Z}\boldsymbol{P}_k^{(i)}\boldsymbol{\Psi}_l(ax-k),\ \tilde{\boldsymbol{\Psi}}_{al+i}=\sum_{k\in Z}\tilde{\boldsymbol{P}}_k^{(i)}\tilde{\boldsymbol{\Psi}}_l(ax-k)\tag{1}$$

The matrices $\boldsymbol{P}^{(i)}(z)$ and $\tilde{\boldsymbol{P}}^{(i)}(z)$ are called matrix symbols of $\{\boldsymbol{P}_k^{(i)}\}$ and $\{\tilde{\boldsymbol{P}}_k^{(i)}\}$, where

$$\boldsymbol{P}^{(i)}(z)=\frac{1}{a}\sum_{k\in Z}\boldsymbol{P}_k^{(i)}z^k,\ \tilde{\boldsymbol{P}}^{(i)}(z)=\frac{1}{a}\sum_{k\in Z}\tilde{\boldsymbol{P}}_k^{(i)}z^k,\ z=e^{i\omega/a}$$

Any $n\in Z$ can be expressed by $n=\sum_{j=1}^{\infty}\varepsilon_j a^{j-1}$, $\varepsilon_j\in\{0,1,\cdots,a-1\}$, We can prove the frequency field can be divided into tinier ones with wavelet packets of this paper and it can enlarge the appearance of high frequency[10].

2 Algorithms of Decomposition and Reconstruction

In this section, we give algorithms of decomposition and reconstruction. It can decompose signals into tinier levels and reconstruct them by using wavelet packets of this paper. It is important in application.

Theorem If $\{\boldsymbol{\Psi}_n\}_{n\in Z_+}$ and $\{\tilde{\boldsymbol{\Psi}}_n\}_{n\in Z_+}$ are biorthogonal wavelet packets given by Eq. (1), then for all $k\in Z$,

$$\boldsymbol{\Psi}_n(ax-k)=\frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\tilde{\boldsymbol{P}}_{k-al}^{(j)*}\boldsymbol{\Psi}_{an+j}(x-l),\ \tilde{\boldsymbol{\Psi}}_n(ax-k)=\frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\boldsymbol{P}_{k-al}^{(j)*}\tilde{\boldsymbol{\Psi}}_{an+j}(x-l)\tag{2}$$

Proof　We have:

$$\frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\tilde{\boldsymbol{P}}_{k-al}^{(j)*}\boldsymbol{\Psi}_{an+j}(x-l)=\frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\tilde{\boldsymbol{P}}_{k-al}^{(j)*}\sum_{m\in Z}\boldsymbol{P}_m^{(j)}\boldsymbol{\Psi}_{an+j}(ax-al-m)=\frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\sum_{m\in Z}\tilde{\boldsymbol{P}}_{k-al}^{(j)*}\boldsymbol{P}_m^{(j)}\boldsymbol{\Psi}_{an+j}(ax-al-m)=$$

$$\frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\sum_{r\in Z}\tilde{\boldsymbol{P}}_{k-al}^{(j)*}\boldsymbol{P}_{r-al}^{(j)}\boldsymbol{\Psi}_{an+j}(ax-r)=\frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{r\in Z}\boldsymbol{\Psi}_n(ax-r)\sum_{l\in Z}\tilde{\boldsymbol{P}}_{k-al}^{(j)*}\boldsymbol{P}_{r-al}^{(j)}=\boldsymbol{\Psi}_n(ax-k)$$

We get first part of Eq. (2), so with second part of Eq.(2).

Given level N, we consider $f\approx f_N:=\sum_{j\in Z}C_j\boldsymbol{\Psi}_0(z^Nx-j)\in V_N$, where $\{C_j\}$ is constant vector sequence with dimension t. But $V_N=W_{N-1}\dotplus V_{N-1}=\cdots=W_{N-1}\dotplus W_{N-2}\dotplus\cdots\dotplus W_{N-M}\dotplus V_{N-M}$, so $f_N=g_{N-1}+g_{N-2}+\cdots+g_{N-M}+f_{N-M}$, where $f_{N-M}\in V_{N-M}$ and $g_j\in W_j$, $j=N-M,\cdots,N-1$.

Let $f_j(x)=\sum_{k\in Z}C_k^j\boldsymbol{\Psi}_0(a^jx-k)$, $g_j(x)=\sum_{i=1}^{a-1}\sum_{k\in Z}D_k^{i,j}\boldsymbol{\Psi}_0(a^jx-k)$,where $\{C_k^j\}_{k\in Z},\{D_k^{i,j}\}_{k\in Z}$, $i=1,2,\cdots,a-1$, $j=N-M,\cdots,N-1$ are constant vectors with dimension t. By using Eq. (2), we can decompose $g_j(x)$ as

$$g_j(x)=\sum_{i=1}^{a-1}\sum_{k\in Z}D_k^{i,j}\boldsymbol{\Psi}_i(a^jx-k)=\sum_{i=1}^{a-1}\sum_{k\in Z}D_k^{i,j}\sum_{m=0}^{a-1}\sum_{l\in Z}\boldsymbol{P}_{k-al}^{(m)*}\boldsymbol{\Psi}_{ai+m}(a^{j-1}x-l)=$$

$$\sum_{i=0}^{a^2-1}\sum_{l\in Z}\Big(\sum_{k\in Z}D_k^{\lfloor\frac{i}{a}\rfloor,j}\boldsymbol{P}_{k-al}^{(i-\lfloor\frac{i}{a}\rfloor)*}\Big)\boldsymbol{\Psi}_i(a^{j-1}x-l)=\sum_{i=0}^{a^2-1}\sum_{l\in Z}D_1^{i,j,1}\boldsymbol{\Psi}_i(a^{j-1}x-l)=\cdots=\sum_{i=0}^{a^{m+1}-1}\sum_{l\in Z}D_l^{i,j,m}\boldsymbol{\Psi}_i(a^{j-m}x-l)$$

where

$$D_l^{i,j,m}=\sum_{k\in Z}D_k^{\lfloor\frac{i}{a}\rfloor,j,m-1}\boldsymbol{P}_{k-al}^{(i-\lfloor\frac{i}{a}\rfloor)*},\quad D_l^{i,j,0}=D_l^{i,j}\tag{3}$$

On the other hand, we can reconstruct $g_j(x)$ as

$$g_j(x)=\sum_{i=0}^{a^{m+1}-1}\sum_{l\in Z}D_l^{i,j,m}\boldsymbol{\Psi}_i(a^{j-m}x-l)=\sum_{i=0}^{a^{m+1}-1}\sum_{l\in Z}D_l^{i,j,m}\sum_{k\in Z}\boldsymbol{P}_k^{(i-\lfloor\frac{i}{a}\rfloor)}\boldsymbol{\Psi}_{\lfloor\frac{i}{a}\rfloor}(a^{j-m+1}x-k)=$$

$$\sum_{i=0}^{a^m-1}\sum_{k\in Z}D_k^{i,j,m-1}\boldsymbol{\Psi}_i(a^{j-m+1}x-k)=\cdots=\sum_{i=1}^{a-1}\sum_{k\in Z}D_k^{i,j}\boldsymbol{\Psi}_i(a^jx-k)$$

where

$$D_k^{i,j,m-1}=\sum_{n=0}^{a-1}\sum_{l\in Z}D_l^{i,j,m}\boldsymbol{P}_k^{(n)}\tag{4}$$

Now, we get Eqs. (3) and (4) as the formulae of decomposition and reconstruction of signals with biorthogonal multiwavelet packets with scale=a. They are practical and can be realized easily.

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